Solutions to End of Course Problem Set
Math 896 Experimentation, Conjecture, Reasoning
Spring 2005

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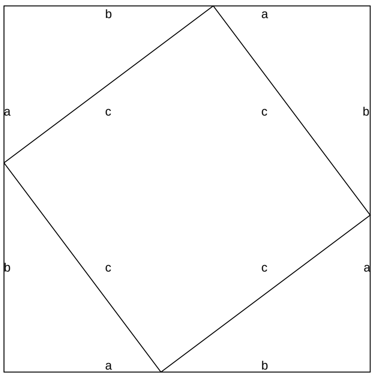
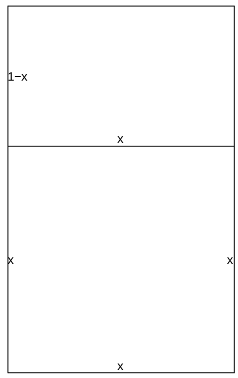
1. From the diagram, equating areas expressed in two different ways, as the outer area of the square on the left, and then the inner area of the four triangles and the inner square on the right:



$$
\begin{aligned}
(a+b)^2 &= 4(ab/2) + c^2 \\
a^2 + 2ab + b^2 &= 2ab + c^2 \\
a^2 + b^2 &= c^2
\end{aligned}
$$

2. My personal preference is for the proof presented above. I find that for me it is easier to construct the figure, and remember the relationships.

3. From the proportions in the figure, you can see that



$$\begin{aligned} \frac{1}{x} &= \frac{x}{1-x} \\ x^2 &= 1-x \\ x^2 - x + 1 &= 0 \\ x &= \frac{-1+\sqrt{5}}{2} \end{aligned}$$

You can check the answer by substituting it in the proportions and rationalizing all denominators.

4.

Solid	Vertices	Edges	Faces	$V - E + F$
Tetrahedron	12	18	8	2
Cube	24	36	14	2
Octahedron	24	36	14	2
Dodecahedron	60	90	32	2
Icosahedron	60	90	32	2

For the tetrahedron, a slice creates a new triangle at each vertex. So each vertex creates 3 new edges. There where 4 original faces, now there are 4 hexagons (one per face) and 4 triangles (one per vertex). There were 6 original edges, and there are 3 new edges per vertex for 12 new edges and a total of 18 edges. Each old vertex becomes 3 new vertices, for a total of 12 vertices.

For the cube, each slice creates a triangle in the corner. SO each vertex creates three new edges. Now there are 6 octagon faces and 8 triangle faces, for a total of 14 faces. There were 12 old edges and 3 new edges for each of the 8 vertices, for a total of $12 + 3 \cdot 8 = 36$ edges. The original 8 vertices become $3 \cdot 8 = 24$ vertices.

For the icosahedron, to the original 20 hexagon faces are now added 12 new pentagon faces for a total of 32. To the t30 original edges are added $5 \cdot 12$ new edges. Each of the 12 original vertices now has become 5 vertices for a total of $5 \cdot 12 = 60$ vertices.

For the dodecahedron, there were 12 old pentagon faces, which after slicing are expanded to 20 additional triangular faces for a total of 32 faces. To the original 30 old edges are added $3 \cdot 20$ edges fro a total of 90 edges. Each vertex is expanded to three vertices , to make a total of 60 vertices.

5. Make a table to make the relationships clear

n	F_n	$5F_n^2 + 4(-1)^n$	
1	1	$5(1^2) + 4(-1) = 1$	1^2
2	1	$5(1^2) + 4(+1) = 9$	3^2
3	2	$5(2^2) + 4(-1) = 16$	4^2
4	3	$5(3^2) + 4(+1) = 49$	7^2
5	5	$5(5^2) + 4(-1) = 121$	11^2
6	8	$5(8^2) + 4(+1) = 324$	18^2
7	13	$5(13^2) + 4(-1) = 821$	29^2
8	21	$5(21^2) + 4(+1) = 2209$	47^2

It seems to be pretty clear that the sequence $5F_n^2 + 4(-1)^n$ is a sequence of perfect squares. What is even more interesting and maybe even amazing is the sequence of integers which when squared give the perfect squares are themselves a new generalized Fibonacci sequence, starting with 1 and 3 and proceeding as sums of the previous two values!

6. The terms of the sequence are 3, 2, 5, 7, 12, 19, 31, 50, 81, 131, 212, 343, 555 .... Let G_n denote the terms of this sequence of numbers. Because

$$\begin{aligned} G_n &= F_n + L_n \\ &= F_{n-1} + F_{n-2} + L_{n-1} + L_{n-2} \\ &= F_{n-1} + L_{n-1} + F_{n-2} + L_{n-2} \\ &= G_{n-1} + G_{n-2} \end{aligned}$$

we could consider this to be a generalized Lucas-Fibonacci sequence with starting values 3 and 2.

Alternatively, use the fact that $L_{n+1} = F_{n+1} + F_{n-1}$, see problem I.7, page 58 in the text. Equivalently, $L_n = F_n + F_{n-2}$ Then

$$G_n = F_n + L_n = F_n + F_n + F_{n-2} = 2F_n + F_{n-2}$$

This is probably about as simple a solution formula as you are likely to get.

7. (a) The probability of having all boys in a family of 5 children is $(1/2)^5 = 1/32 = 0.03125$, assuming independence of birth of boys and girls, as well as equality of probability

   (b) The probability of at least one girl is the complement of the probability of *all* boys. So to have the probability of at least one girl in a family of n children is $1 - (1/2)^n$. To have this probability be greater than 95% we must have the complementary probability of all boys be less than 5%. A family of four boys, occurring with probability $(1/2)^4 = 1/16 = 0.0625$, is still too big, while the probability of 5 boys computed above is: $0.03125 < 0.05$. So five children is the required number to have a probability of at least 95% of at least one girl.

8. There are 35 possible problems from each Section. The number of ways the test can be made up is:

$$\frac{(35 \cdot 34)}{2 \cdot 1} \cdot \frac{(35 \cdot 34)}{2 \cdot 1} \cdot \frac{(35 \cdot 34)}{2 \cdot 1} \frac{(35 \cdot 34 \cdot 33)}{3 \cdot 2 \cdot 1} \cdot \frac{(35 \cdot 34 \cdot 33)}{3 \cdot 2 \cdot 1} \cdot \frac{(35 \cdot 34 \cdot 33)}{3 \cdot 2 \cdot 1}$$

The probability that this specific question will be on the test is:

$$\frac{(35 \cdot 34) \cdot (35 \cdot 34) \cdot (35 \cdot 34) \cdot (1 \cdot 34 \cdot 33) \cdot (35 \cdot 34 \cdot 33) \cdot (35 \cdot 34 \cdot 33)}{(35 \cdot 34) \cdot (35 \cdot 34) \cdot (35 \cdot 34) \cdot (35 \cdot 34 \cdot 33) \cdot (35 \cdot 34 \cdot 33) \cdot (35 \cdot 34 \cdot 33)} = \frac{1}{35} \approx 0.02857$$

9. (a) The expected value of this insurance policy is:

$$(5000 - 150) \cdot 0.02 + (200 - 150) \cdot 0.10 + (-150) \cdot 0.88 = -30$$

(b) The fair price of the policy is determined by:

$$(5000 - P) \cdot 0.02 + (200 - P) \cdot 0.10 + (-P) \cdot 0.88 = 0$$

Solving, $P = 120$

10. Imagine that the absent-minded professor answers the question honestly but absent-mindedly for each of the 100,000 students. Then theoretically the professor answers correctly that 180 of the scholarship students have a scholarship and falsely that 20 of the scholarship students have do not have a scholarship. The professor answers correctly for 89,820 students that they do not have a scholarship. Finally, the professor answers falsely for 9980 students that they do have a scholarship. So if you ask the professor, and he answers that you do have a scholarship, the chances are $180/(180 + 9980) = 0.0177$ that you really do have a scholarship. By consulting an only partially reliable source, you raise your estimate of having a scholarship from 0.002 to 0.01777, not quite an order of 10.

Here's another way to do the problem with Bayes' Formula. Let S be the event of winning a scholarship and PY that the professor says yes, you do have a scholarship. Then

$$\begin{aligned} P[S|PY] &= \frac{P[PY|S] \cdot P[S]}{P[PY|S] \cdot P[S] + P[PY|\text{not}S] \cdot P[\text{not}S]} \\ &= 0.9 \times 0.0020.9 \times 0.002 + 0.1 \cdot 0.998 = 0.0177 \end{aligned}$$