

Question of the Day

What is the probability that in a family with two children, both are boys? What is the probability that in a family with two children, both are boys, if we already know that one is a boy? What changes if we know that the older child is a boy? What makes the second and third probability questions different from the first? (Assume that the probability of having a boy is $1/2$, and the sex of each child is independent from the sex of other children.)

Key Concepts

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1. Frequently we wish to compute the probability of an event but have additional information that may alter the sample space. This means that rather than simply computing $\Pr[E]$, we re-evaluate $\Pr[E]$ in light of the information that F has occurred, that is, we consider the altered sample space.

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$$

- 2.
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Vocabulary

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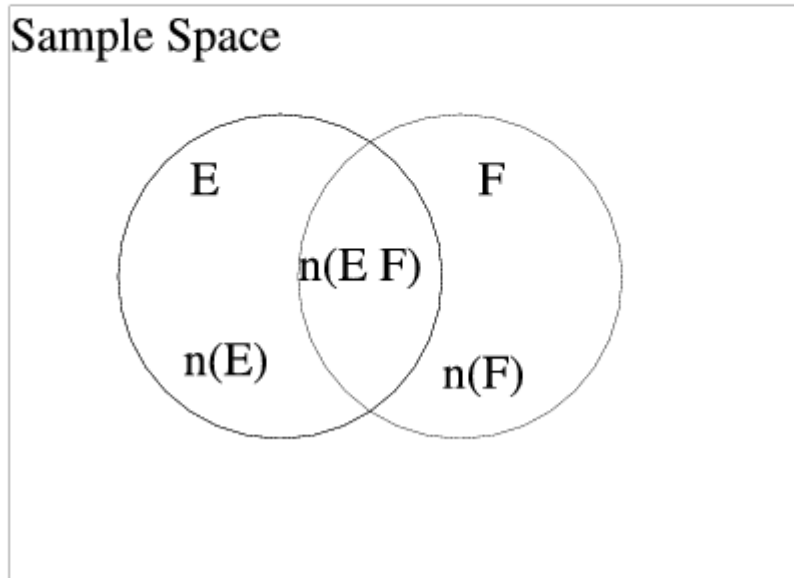
1. The *conditional probability* of an event E given that an event F has occurred is denoted $\Pr[E|F]$.
 - 2.
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Mathematical Ideas

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Conditional Probability with Venn Diagrams

Consider the Venn diagram for two events E and F shown in the figure



We seek $\Pr[E|F]$.
 Let $n(E)$, $n(F)$, and $n(E \cap F)$ be the number of ways that events E , F and $E \cap F$ can occur among all the outcomes of the experiment. Now knowing the additional information that event F has occurred, we find $\Pr[E|F]$ by focusing our attention on the region in the event circle labeled F . We evaluate the number

$$\frac{n(E \cap F)}{n(F)}$$

which represents the number of times that E has occurred among all those outcomes in which F has occurred. Now

$$\frac{n(E \cap F)}{n(F)} = \frac{\frac{n(E \cap F)}{n}}{\frac{n(F)}{n}} = \frac{\Pr(E \cap F)}{\Pr(F)}$$

Sometimes it is easy to calculate the conditional probability, but may be hard or confusing to compute the probability of the intersection. In such a case, we can turn the conditional probability formula around by multiplying through by the denominator and obtaining:

$$\Pr[E \cap F] = \Pr[E|F] \Pr[F].$$

A Sample Problem

Suppose I have rolled two dice and I announce that the sum of the two dice is 5 or less. Compute the probability that the sum of the two dice is an even number.

Solution: First I list all the possible ways to roll two dice and have a sum less than or equal to 5:

Dice 1	Dice 2	Sum
1	1	2
1	2	3
2	1	3
1	3	4
3	1	4
2	2	4
1	4	5
4	1	5
2	3	5
3	2	5

We see that there are 10 possible outcomes. We are interested in how many ways to roll to get an even number in the sum column. Observe that 2 and 4 are the only even numbers less than 5. There is one way to roll the dice and have their sum be 2. There are three ways to roll the dice and have their sum be 4. Hence, the probability is $\frac{4}{10}$.

Sampling without replacement

Suppose an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that as we draw the each ball in the urn is equally likely to be drawn, what is the probability that both balls drawn are red?

Let R_1 and R_2 denote respectively the events that the first and the second ball drawn is red. Now given that the first ball selected is red, there 7 remaining red balls and 4 white balls and so $\Pr[R_2|R_1] = 7/11$. As $\Pr[R_1] = 8/12$, the desired probability is

$$\Pr[R_1 \cap R_2] = \Pr[R_1] \Pr[R_2|R_1] = (8/12)(7/11) = 14/33.$$

Note that another way to work this problem is use combinations by thinking of drawing a pair of balls in quick succession:

$$\Pr[R_1 \cap R_2] = \frac{\binom{8}{2}}{\binom{12}{2}}.$$

Extended Sampling without replacement

An urn contains 6 white and 9 black balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and the last 2 selected are black?

Let the event that the first ball drawn is white be W_1 , the second ball selected is white be W_2 , the third ball selected is black be B_3 and the fourth ball selected is black is B_4 . The we are interested in the probability of $\Pr[B_4B_3W_2W_1]$ Now

$$\Pr[B_4B_3W_2W_1] = \Pr[B_4|B_3W_2W_1] \Pr[B_3W_2W_1]$$

But further decomposing

$$\Pr[B_3W_2W_1] = \Pr[B_3|W_2W_1] \Pr[W_2W_1]$$

and as in the previous example:

$$\Pr[W_1 \cap W_2] = \Pr[W_1] \Pr[W_2|W_1].$$

Putting them altogether:

$$\Pr[B_4B_3W_2W_1] = \Pr[B_4|B_3W_2W_1] \Pr[B_3|W_2W_1] \Pr[W_2|W_1] \Pr[W_1].$$

Fortunately, each conditional probability is easy to compute:

$$\Pr[B_4B_3W_2W_1] = \frac{8}{12} \frac{9}{13} \frac{5}{14} \frac{6}{15}$$

This section is adapted from: *Finite Mathematics*, by Karl J. Smith, Scott, Foresman and Company, Glenview, IL, 1975. Some of the problems and examples are from *A First Course in Probability*, by Sheldon Ross.

Problems to Work for Understanding

1. Suppose a family has four children.
 - a. What is the sample space?
 - b. What is the probability of 4 boys? 4 girls?
 - c. What is the probability of 1 girl and three boys? 1 boy and three girls?
 - d. What is the probability of 2 boys and 2 girls?
 - e. What is the sum of your answers in parts b through d?
2. What is the probability of a family of three children consisting of two boys given that at least one of them is a boy? (Assume the probability of a boy child is the same as a girl child.)
3. A couple has two children. What is the probability that both are girls if the eldest is a girl? (Assume the probability of a boy child is the same as a girl child.)
4. What is the probability of flipping a coin four times and obtaining three heads, given at least two are heads?
5. A chairman and a vice-chairman of a board consisting of 10 men and 5 women are to be chosen. After

- a chairman is chosen, what is the probability that the vice-chairman is a woman?
6. Two fair dice are rolled. What is the conditional probability that the first one lands on 6 given that the dice land on different numbers?
 7. What is the probability that at least one of a pair of fair dice lands on 6 given that the sum of the dice is j , $j = 2, 3, \dots, 12$.
 8. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade, given that the second and third cards are spades.
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