

Experimentation, Conjecture and Reasoning for Middle Level Teachers

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Description

The goal for this course is to develop the students' mathematical habits of mind. A productive mathematical thinker has a toolbox of skills and knowledge to experiment, conjecture, reason, and ultimately solve problems. "Mathematical habits of mind" means both understanding of and experience using these tools. Mathematical habits of mind are marked by great flexibility of thinking and precise exposition of solutions are important. Flexibility of thinking includes the use of indirect arguments as well as making connections between knowledge the mathematical thinker possesses and the problem being considered.

Although a complete mathematical toolbox includes algorithms, a person with well developed habits of mind knows both why algorithms work and under what circumstances an algorithm will be most effective. Mathematical habits of mind are also marked by ease of calculation and estimation as well as persistence in pursuing solutions to problems. A person with well developed habits of mind has a disposition to analyze all situations as well as the self-efficacy to believe that he or she can make progress toward a solution. Such a person also engages in metacognition by monitoring and reflecting on the processes of reasoning, conjecturing, proving, and problem solving.

Some of the materials used to teach the course and some additional material is available online at <http://www.math.unl.edu/~sdunbar1/ExperimentationCR/experimenta>

Instructional Style

The *Math in the Middle* curriculum has included Math 804T, *Experimentation, Conjecture, and Reasoning* four times since 2005, 3 of those times in the fall semester as the third course in the curriculum. Since the course was in the fall semester and the teacher/students were back in the classroom in their locales, the course was taught as a distance-education course with the Internet as a communication mode. The course used the BlackBoard classroom content management system for course documents, announcements, information, student discussion and question and answer. The course used the EDU on-line course and homework system (now known as MapleTA) for administering and scoring bi-weekly quizzes. The course used Breeze (now known as Adobe Connect) for group tele-conferencing on a weekly basis. The text was *The Heart of Mathematics* by Burger and Starbird.

The course divided the semester into four generally independent sessions. Three of the four sessions were about 3 weeks in length, and had two sub-sessions, each 1 to 2 weeks in length. After teaching the course two times, we extended the session on Geometry to have 5 sub-sessions, each about a week in length. In most sub-sessions students completed a regular pattern of activities:

- Read the relevant material, mostly from the text-book. We supplemented the Geometry and Probability sections with additional reading and study material.
- Complete the Pre-Test quiz on EDU. The Pre-Test consisted of 5 mathematics problems related to the topic of the sub-session. The problems were drawn from prior years' MAA American Mathematics Completions (AMC) contest problem sets.
- Complete 8 problems from the text and submit them for grading by faxing them. The first five problems exercise basic calculation and reasoning skills. The next 2 problems exercise more challenging calculation and reasoning skills as well as persistence in pursuing solutions. The final problem exercises flexibility of thinking as well as making connections between knowledge and the problem.
- A Post-Test quiz from EDU, consisting of another 5 mathematics problems drawn from prior years' AMC contest problem sets.

At the end of the semester, the students completed an End of Course Problem set as a comprehensive take-home final on the entire content of the course.

The course also included a Learning and Teaching Project. For the Learning and Teaching Project each semester we required all the students to use two problems on the Fibonacci sequence from the text as the basis for the Learning and Teaching lesson. We found these two problems were adaptable to a wide range of student ages and courses.

The course started with a weekend workshop in early September and continued through the end of the academic semester when the End of Course Problem set was due. The following course had a weekend workshop in January 2008 at which time we returned and discussed all the graded Teaching and Learning problems and final grades for the course.

For the first couple of years the instructor conducted the course in a fairly straightforward fashion where the students read the text, then worked the problems at the end of the Chapter. In weekly on-line problem solving sessions conducted with through Breeze (now Adobe Connect) teleconferencing software, the students discussed solving the problems among themselves with the aid of the instructor or a teaching assistant. However, as the course developed through several years, the instructor developed a different style of problem presentation and solution which suited the nature of the course better.

In the third year of the course the instructor introduced a new module in the Geometry section of the course centered on problems in classical Euclidean geometry employing GeoGebra software. The problems were taken from prior American Mathematics Competitions contests. For each problem the students did a construction of the problem with GeoGebra first. Then the students measured the solution numerically to a couple of decimal places using the algebraic part of GeoGebra. The students then made a conjecture about the solution of the problem. The conjecture usually expressed a geometric construct such as a length or area using mathematical constants. The students then explicitly solved the problems symbolically, using similarity or area formulas. This structure turned out to be an excellent way to organize the problems. This pattern fit exactly with the title and goal of the course to be experimentation, conjecture and reasoning. The reasoning usually took the form of explicitly solving the problem, and expressing the answer analytically or symbolically, rather than a logic proof. The students are ready to reason and explain a solution of a problem, but do not yet have the logic

tools, experience and enough examples to structure a mathematical proof.

This pattern of software experiment, conjecture, and then reasoning out a symbolic solution turned out to be an excellent way to structure the exercises of the course. A goal for future versions of the course is to re-structure all of the exercises from the text with extensive, structured and directed experimentation with software. For the geometry sections, GeoGebra is an excellent choice. For the sequences and counting section, calculators or spreadsheet software would be good choices for computer experimentation. Then the re-structured problems would explicitly ask students to make a conjecture about the final answer with symbolic constants involving π , square roots, Golden ratio, combinatorial functions etc. Then finally, the re-structured problem would ask for a formal derivation, say by setting up and solving an equation to derive the symbolic answer. Each part will allow for partial credit. Another idea that recognizes the current style of student research and exploration would be to additionally require the students to do a Web search on some problems to find resources or applets that extend, illustrate or solve the problem. More points would be awarded for finding resources at a trusted site such as the NCTM Illustrations, or mathworld.com. Another restructuring goal is to structure problems with a worksheet format, so the students fill out the worksheet in a systematic way. The first few Getting Started problems would be highly structured, the second group of New Ideas problems would somewhat less structured, and then finally, the Habits of Mind problem would be just simply stated, not structured at all.

A beginning on this form of problem solving occurs in the Fall 2007 End of Course Problem set included in this article in the Appendix. A few of the problems count points for some experiments and a conjecture, and then additional points for the full solution to a symbolic answer. The new idea would be to expand this grading scheme to a larger number of points for the experiment, even breaking the experiment into several parts. Then the conjecture would be some additional points, depending on how it was expressed. Then finally the symbolic solution would be worth some final points.

Course Outline

Beginning Workshop To “Experience ideas in as many ways as possible”, to “Take ideas from one domain and explore them in another” and

“Understand simple things deeply”.

Session A: Numeric Patterns To learn to “Look for patterns”, “Find hidden, underlying structure” and “Understand simple things deeply”.

1. Fibonacci Numbers I (Burger and Starbird, Chapter 2.2, pages 49-63)
2. Fibonacci Numbers II (Additional material)

Session B: Geometry To “Experience ideas in as many ways as possible”, to “Take ideas from one domain and explore them in another” and “Understand simple things deeply”.

1. Pythagorean Theorem (Burger and Starbird, Chapter 4.1, Pages 208-217)
2. The Golden Rectangle (Burger and Starbird, Chapter 4.3, pages 232-247)
3. The Platonic Solids (Burger and Starbird, Chapter 4.5, pages 269-288)
4. GeoGebra Session I (Additional Material)
5. GeoGebra Session II (Additional Material)

Session C: Counting and Probability, “Simple clear cases let us apply principles that we can apply widely.”

1. General Counting (Burger and Starbird, Chapter 7.4, pages 554-567)
2. Probability (Burger and Starbird, Chapter 7.2, pages 523-540)

Session D: Conditional Probability, “Make it Quantitative” and “Beware of unintended consequences.”

1. Conditional Probability (Additional Material)
2. Bayes Theorem (Burger and Starbird, Chapter 8.2, pages 645-662)

End of Course Review, A comprehensive look back at the course.

1. End of course review.

Course Materials

The primary text is selected readings and exercises from *The Heart of Mathematics, Second Edition*, by E. Burger and M. Starbird, Key College Publishing, 2005 (ISBN 1-931914-41-9) and the corresponding activities kit. Some supplementary additional readings were posted on the course website. Some additional exercises were adapted from the MAAs American Mathematics Competitions contests, especially the middle-school level AMC 8 contest.

Online Resources

Most of the course materials are available at: www.math.unl.edu/~sdunbar1/Teaching/Experi

Acknowledgements

The course developer is Steven Dunbar, Professor of Mathematics at the University of Nebraska-Lincoln. Prof. Dunbar has taught the course a total of five times, four times for the Math in the Middle Project and once as a distance course offered by the Department of Mathematics. The course was inspired by a suggestion from Prof. Jim Lewis who had heard of a similar course. The course was also inspired by the contents and presentation style of the course textbook by Burger and Starbird. Dunbar's interest in teaching and learning mathematical problem solving guided the course development.

Connections to the middle level curriculum and concrete examples

This project explores learning and teaching from two perspectives. How can you embed the mathematics that you learn in your classroom? Does there exist a relationship between how well you understand the mathematics and how successful you can be as a teacher of that mathematics? For the Teaching and Learning Project each semester we asked each teacher/student to use problems 6 (Baby Bunnies) on page 58 and 9 (Late Bloomers) on page 59 of Burger and Starbird as the basis for a lesson. These two problems are the original context for the Fibonacci sequence along with a small generalization.

We found these topics were adaptable to a wide range of student ages and courses. We asked each student/teacher make the judgment about where to start with their students, what each teacher/student expected from their students, and how far each could push the students. More details about what to do in the Teaching and Learning Project on the page titled “Math 804T: Experimentation, Conjecture, and Reasoning, Math in the Middle, Learning and Teaching Project”. The materials each teacher/student turned in for the Teaching and Learning Project was assessed according to the rubric outlined on the pages labeled “Assessment Rubric for Written Reflections on Learning and Teaching Project”. Both documents are available on the course web site.

The Learning and Teaching Project was valuable in encouraging the teacher/students to apply the mathematics that they have learned. Many teacher/students approached the lesson with some fear that the material would not be adaptable to the classroom, or would interfere with required teaching material. Most found it to be a valuable teaching experience and almost all of them learned something about themselves, their students and their teaching style.

Appendix I: Sample Problem Set

End of Course Problem Set Math 804T

Experimentation, Conjecture and Reasoning Steven R. Dunbar Fall Semester 2007

1. Consider the sequence of Lucas numbers L_n , defined in Section 2.2, page 59, problem 10, and the sequence of Fibonacci numbers F_n . For 2 points, can you conjecture what the ratio L_n/F_n seems to approach as n gets larger? For 3 points can you provide some mathematical reasoning that justifies your conjecture with the methods of Section 2.2? (Hint: Use the result of Problem 12, page 59 and some of the ideas in of Problem 29, page 61.)
2. Make a new sequence S_n which is the sum of the first n Fibonacci numbers. For 2 points, make a table of values of S_n and make a conjecture about expressing this sequence in terms of the Fibonacci sequence. For 3 points, provide mathematical reasoning about why your conjecture is true.

3. Can you make a triangle with the following leg lengths: one leg is the sum of two consecutive odd integers, the other leg is the product of two consecutive odd integers, and the hypotenuse is two more than the product? Can you make a right triangle with those leg lengths? For two points, provide some experimentation and a conjecture, for 3 points, provide some mathematical reasoning which justifies your conjecture.
4. The cube can be inscribed in a sphere, that is, a sphere which touches the 8 vertices of the cube. What is the radius of that sphere which has an inscribed cube of side length 1? (For bonus points, not required, just for fun and extra points if you get it right! What is the radius of a sphere which has an inscribed tetrahedron of side length 1?)
5. A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of the circle? (Two points for a decimal approximation correct to two places, 3 points for mathematical reasoning that gives a mathematical answer.)
6. At a teaching council 4 math teachers, 3 English teachers and 3 foreign language teachers are to be seated in a row. How many seating arrangements are possible when teachers of the same subject are required to sit together?
7. A mother is holding a birthday party with several excited young children. She has n distinctively wrapped party favors to give to n children. She now has a headache, so she quickly hands out each favor package randomly without looking to see if the recipient already has a package. For $n = 3, 4,$ and $5,$ find the probability that each child gets a package.
8. A paper bag discovered at the back of a closet shelf is found to contain twelve old light bulbs. Five of them are 25-watt, six are burned out and three are both. Find the probability that a bulb drawn at random from the bag is burned out, given that it is 25-watt. Also find the probability that it is not 25-watt given that it is burned out.
9. Suppose that an insurance company classifies people into one of three classes – good risks, average risks and bad risks. The company records indicate that probabilities that good, average and bad risk persons will be involved in an accident over a 1-year span are respectively 0.05, 0.15

and 0.30. If 20% of the population are good risks, 50% are average risks and 30% are bad risks, what proportion of people have accidents in a given calendar year? If a policy holder Steve had no accidents in 2007, what is the probability that Steve is a good risk?

10. Write a short (one page or less) reflection on what was your favorite problem in the course, and why it was your favorite. Explain what you learned about mathematics and problem solving from your favorite problem.