How to Focus the Mathematics Curriculum on Solving Problems

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What does it mean to have a mathematics curriculum that focuses on problem solving? At one time, many of us thought that a “problem solving” focus required unusual, unfamiliar, contextualized problems that looked entirely different from conventional classroom problems. These problems were often entertaining and interesting and sometimes dealt with important mathematics—but both students and teachers viewed them as completely separate from the rest of the mathematics curriculum. This chapter presents a problem-solving approach in a different sense—the use of problems to focus on the coherent development of important mathematical ideas that are core to the curriculum. As the examples in this chapter reveal, problems that engage students in reasoning about, representing, explaining, and reevaluating their ideas can be surprisingly straightforward. The choice of particular problems is not what defines a problem-solving curriculum. Rather, it is created through a coherent focus on mathematical ideas.

In this chapter, we recount episodes from four classrooms in which teachers made decisions about how to engage students with
significant mathematical ideas. These stories illustrate how the interaction of students, teacher, and curriculum content can create a learning environment in which students' thinking about fundamental mathematical ideas is the focus of classroom activity and discourse.

The teachers in these episodes were participants in a small study group focused on the partnership of teacher and curriculum materials. The study group met for one year to discuss how to teach effectively using curriculum materials as a core but without considering the materials to be a script or a recipe. What characterizes a classroom in which a teacher is actively engaged in using and modifying such materials? What issues and decisions do teachers face? As part of their work, teachers wrote regularly about classroom episodes that illustrated such issues and decisions.

The four sections of this chapter illustrate four interrelated facets of using or modifying the curriculum to focus on solving problems:

- focusing on fundamental mathematical ideas,
- modifying problems to respond to students' evolving understanding,
- meeting a range of student needs, and
- developing representations and metaphors.

Episode 1, "Why Do You Always Call a Half a Half When Sometimes It's a Whole?"

Focusing on Fundamental Mathematical Ideas

Teaching mathematics involves identifying important mathematical ideas for a particular grade or age, recognizing the ways in which these complex ideas develop over time, and listening and probing for students' developing understanding of these ideas. No matter how well a curriculum is structured to highlight significant mathematical ideas, the teacher's task is to decide, for a particular class, when and how to focus the students' attention on an idea that is central to their learning. Sometimes these ideas emerge in ways the teacher anticipated as she or he planned the lesson. At other times, ideas arise unexpectedly from students' thinking, and the teacher has to decide whether to pursue them. Lucy, a fourth-grade teacher, recounts the following story about how a student in her class raised an important issue about fractions:
My fourth-grade class was about two weeks into a curriculum unit on fractions. We had been working fairly successfully, I thought, breaking squares into halves, fourths, eighths, thirds, sixths, and other fractional parts in a variety of ways. The students seemed to recognize how shapes that looked different could be equivalent fractions, and they were developing an understanding of common equivalents, such as $1/2 = 2/4$, $3/6 = 1/2$, and $3/4 = 6/8$. I thought things were going well when, during a class discussion, Jessie, looking very perplexed, asked a question that had been bothering her for several days: "Why do you always call a half a half when sometimes it’s a whole?" Other students protested, "Because it’s not a whole, it’s a half," "What are you talking about? You know what a half is—when you cut something into two pieces, you get a half, and there’s another half." But Jessie was persistent: "No! When I cut something into two pieces and I take my piece, it’s a whole. Like those brownie problems—when I cut my brownie into two pieces and I take my piece, I get a whole piece."

I sat back for a while and listened to the students interact around Jessie’s question. Some tried to use a pizza example as the basis of their argument: when you buy a pizza, it’s usually cut into 8 pieces; when you take a piece, you are taking 1/8 of the pizza. But Jessie remained unconvinced. This example seemed to be the same as the brownie example she had proposed: "When I take my piece, I’m taking a whole piece." After letting the discussion go on for a while, I asked if others in the class thought about one-half the same way Jessie did. I thought she might be the only one, but then a hand went up and then another and another. Soon, some of the students were saying that they understood what Jessie was saying and were equally confused about it.

We can imagine situations in which Jessie’s question would be brushed aside. After all, most of the students (and all the initially vocal students) seem to understand what is going on—that half of a brownie is still half of the brownie, even when it becomes someone’s whole portion. Before moving the class on to the planned work, the teacher could easily answer Jessie’s question with a simple clarifying statement—for example, "Well, yes, Jessie, if you think of the piece of brownie you get as a whole piece, then it is a whole, but it is still half of the original brownie.” Would this statement be enough to satisfy Jessie and clear up her confusion? Also, the possibility exists that Jessie might never have asked her question. The fact that she brought up her question and persisted in
her argument in the face of immediate disagreement from other students suggests that some things of importance are going on in this classroom—(a) that students know that their ideas are valued and (b) that mathematics is about thinking through ideas, challenging ideas that are being discussed, and developing logical arguments to support an idea or call it into question.

Jessie may or may not be confused, but she has brought up a central idea about fractions—that a fraction is a fraction in relation to a particular unit whole. A quantity is named differently when it is considered in relation to a different unit. Lucy chooses to wait and listen before making a decision about how to intervene. She recognizes several important elements in this situation. First, Jessie is wondering about a central mathematical idea. Second, once they hear Jessie’s argument, several other students have the same question. Lucy may also be considering another common aspect of learning about a complex idea such as fractions: students’ apparent understanding of some aspects of a topic may conceal a lack of understanding of fundamental ideas. The relation of a fraction to a unit whole may seem obvious when the unit has been clearly identified, for example, when students are cutting up squares that represent brownies. But do students have the flexibility to keep track of the mathematical relationships between piece and whole as different units are considered? If I have two slices of the three pizzas we ordered for our meal, with each pizza cut into eight slices, am I eating two whole slices, a quarter of a pizza, or a twelfth of the whole meal?

Here is a moment at which a teacher must make a decision about how to problematize the curriculum. Although a good curriculum can alert teachers to the fundamental mathematical issues that might arise, it cannot possibly anticipate everything that will happen in class—or when it will happen. The teacher must listen to students’ talk and observe their actions, analyzing these discussions and behaviors in relation to the mathematical agenda for the class and making decisions about when to stop and focus on an essential mathematical idea. Lucy, who had been involved in a series of teacher development projects, had been thinking hard about mathematics with other colleagues for several years. She comments on the importance of the teacher’s interaction with the curriculum:

I wonder how I would have handled Jessie’s question three or four years ago. I don’t think that I would have understood the significance of what she was asking. I might have just dismissed the idea
as a minor confusion—something that she would get over as we moved along. I might not have even heard the question. To Jessie’s credit, she did not let the issue rest. She forced us to rethink what we had done over the past weeks, to go back a bit, to retrace our steps, to think of those fractional pieces again and how they relate to the whole.

Although the written curriculum keeps marching along, mathematical ideas do not develop in straight lines or all at once but, rather, in fits and starts, circles and backtracks. A crucial aspect of problematizing the curriculum is taking advantage of moments in students’ thinking that focus—or refocus—the class’s attention on fundamental mathematical ideas.

**Episode 2, “The Counting Jar”: Modifying Problems to Respond to Students’ Evolving Understanding**

A focus on the basic mathematical ideas underlying students’ learning leads to the need to modify core activities to fit students’ evolving understanding of a set of ideas. In the following episode, a kindergarten teacher recounts how she used an activity from the written curriculum materials, then gradually modified it over the course of the year so that it would continue to be problematic for her students. The activity is called “The Counting Jar,” a classroom routine for kindergarten in *Investigations in Number, Data, and Space* (Scott Foresman, 1998), that is done periodically throughout the year. It involves three steps: first, students count the number of objects in the jar, which vary in size and amount from one time the activity is used to the next. Then students represent the number of objects in some way on paper, and finally, they create an equivalent set of objects. When Gloria first read the description of this activity in the curriculum, she noticed that it was similar to another familiar class activity in which students estimate the number of something in a jar, then count. At first she thought that this activity would be pretty much the same, but as she tried it out, she found that it was more appropriate for her students:

By reading carefully the description of the counting jar in the curriculum guide, I became aware of a shift in my thinking. I once
thought that I was providing a challenge for the children by asking them to estimate the quantity in a jar. Their estimation would, I thought, provide a context for practicing counting. I now realize that often either my young children would be frustrated by the estimation or it would seem silly to them—the actual counting seemed meaningless. We would often count the contents of the jar together. I look back now and realize that more times than not I was doing all the counting, with the children chiming along in unison. I used to say that I was modeling ways to count for the children and thought that was enough. Now, with the counting jar, my students are independently counting all the time. As they count, I become more and more aware of their comfort level and proficiency with the process. In this three-step routine, students can utilize the help of others in the first two steps; they can count with a partner to determine the quantity of the original set, and they can look at another child’s paper to see how he or she represented the total, but they have a very hard time avoiding doing the last step—creating an equivalent set of objects—on their own. I feel much more secure that all children are doing the work, and, in fact, I can observe which parts they seek assistance for and which they can do independently.

However, as the year went on, Gloria found that when she brought out the counting jar, some students began complaining, “Do we have to do this again?” She began thinking about how students’ ideas about counting were evolving and how she might modify the counting-jar activity: “I wanted to think of a way to alter the counting jar just a little to see if I could provide a challenge for these students without losing the focus of counting and making it too difficult for others.” She began by increasing the number of objects in the jar. When there were 31 items, rather than 17 or 20, she found that many students miscounted, providing the opportunity for discussing counting strategies, double-checking, and ways to ensure accuracy. In the spring, she introduced another variation:

My newest adjustment to the counting-jar activity is to strategically place subsets of objects in the jar. Last week I put 17 plastic bunnies in the jar—5 were yellow, 5 were red, 5 were blue, and 2 were green. It can take several days for each child to complete the three steps involved in the counting-jar routine. By the fourth day I noticed that everyone had finished. This [completion] usually is my signal to bring the counting jar and recording chart to a class meet-
ing so that we can discuss any new insights and confirm the exact total. The meeting began, and as usual, we acknowledged the representations and confirmed the total. Everyone agreed that it was 17. We looked at how to write the number, because most, if not all, of my students now write the number as opposed to drawing a picture or making tally marks. Reversals are common, and so I use this occasion as a good opportunity to briefly discuss handwriting and strategies for recalling how to write a number.

This meeting had gone smoothly and had not taken very long, so I decided to take on the idea of the subsets even though no one had noticed this aspect on their own. To start, I asked, "Is there anything else that you notice about these bunnies?" After a brief silence, Joseph said, "They are plastic." I nodded and waited a few more seconds. Carly raised her hand next, "They are different colors." I asked, "What do you notice about the colors?" The following conversation ensued:

*Martina:* They are blue and green and yellow and red.

*Kim:* There is the same number of red and blue ones.

*Teacher:* What do you mean?

*Kim:* There are 5 red and 5 blue ones.

*John:* Yeah, and 5 yellow ones.

*Anika:* There aren't five green ones. There are only 2.

*Teacher:* Did anyone notice this before? [There was a long pause. I assumed the silence meant that no one had.] Now that we have noticed how many bunnies are in each color group, can that help us find another way to tell how many bunnies there are altogether? [Again a long pause ensued. Finally Pake spoke.]

*Pake:* I know that 5 and 5 makes 10.

*Kim:* I can count 5, 10, 15, 20.

*John:* No, 17! 5, 10, 15, 16, 17.

*Teacher:* Does everyone follow what John and Kim and Pake just said?

I looked around. Some children were nodding their heads yes, but most were not. We recounted the bunnies together, grouping them by colors although still counting by ones so that everyone could follow. By now we had been sitting for a long time, so I decided to leave the discussion there. I was delighted that some of the children could make use of the subsets and could find a new way to arrive at a total. I also realized that this activity was all done in the support of the whole-group meeting. I look forward with great anticipation to see what will happen the next time I put a similar set of objects in the counting jar.
Gloria was able to use her written curriculum as a starting point, then carefully observe how students interacted with the task and modify the task to meet her students' growing understanding about number. To do so, she had to take into account both what she knew about her students and how the mathematical ideas they were working on can develop. Counting by ones leads to thinking of numbers as being composed of larger units as those ones are combined into groups. Eventually these ideas lead to understanding how our number system is constructed of ones, tens, hundreds, and so forth. For now, some students are beginning to count the objects by using larger chunks—in this instance, fives—and one or two students are starting to coordinate two different units—counting by fives and by ones (“5, 10, 15, 16, 17”) to construct a number.

In reflecting on her decisions, Gloria recognizes both what she learned from the written curriculum and her own role in revising it: “I have learned a great deal by watching my students work with the counting jar and now feel much more secure about making subtle changes. I know that the changes I make are mathematically sound and directly linked with the goals I have set for my students.”

**Episode 3, “The Same Gone and the Same Left”: Meeting a Range of Student Needs**

A crucial aspect of the teacher’s work in problematizing the curriculum for students is taking into account the range of student needs that exist in every classroom. Students who are struggling, students who are doing fairly well but are not going beyond what is expected, students who easily solve most problems, students who can tackle problems beyond what most of their peers are ready for—all these students need problems that engage them in working on significant mathematics. In Gloria’s episode, several students were ready to think about coordinating different units, and Gloria’s modification of the counting-jar activity allowed those students to work on this idea. In the following episode, Margery, a fifth-grade teacher, works with three girls who are struggling in mathematics.

Margery’s class has just completed the first two sections of their fractions unit. As she thinks about how the class is doing as a whole, she identifies three students who seem to be having great difficulty with basic ideas about comparing fractions. Two of the students are bilingual. The third one says she hates mathematics;
Margery describes her as a student who “tries hard not to get involved in any mathematics conversation and attempts to disguise the ‘not knowing’ with angry indifference.” During a time when the rest of the class is writing, Margery meets with the three girls. She begins with a question she frequently asks, encouraging students to take responsibility for their own learning: “What are some ideas about fractions that you don’t completely understand?” No response is forthcoming from the three girls, who look, respectively, sad, scared, and angry. “Clearly,” Margery notes, “these are familiar feelings for them in this difficult mathematics terrain. So many overlays of emotions are evident once one equates ‘not knowing’ with failure.” So Margery asks, “What do you understand?” After a long pause, some conversation begins. Here is Margery’s account of the rest of her meeting with the girls.

_Maria:_ It’s about parts.
_Pauline:_ Parts can be big, or they can be small.
_Maria:_ All the parts in each one have to be exactly the same as each other.
_Cara:_ I don’t know anything. Well, the thing has to be the same size.

The pauses are long here, and I’m trying to figure out what’s the right place to enter, whether these three students share some common ground.

_Cara:_ [with some impatience] It could be about candy bars.
_Pauline:_ We could break them into pieces.
_Teacher:_ Can we think of a context, a story about candy bars and pieces?
_Cara:_ Yeah, we had candy bars and we wanted to eat them, and we didn’t want you to catch us, so we broke them into pieces to sneak them.

I decide to go with this idea. It reminds me of another problem that has proved to be powerful for comparing fractions in my class, a scenario involving loaves of bread sliced into different numbers of slices. We draw four rectangles to represent candy bars, one for each of us. We note that they are all the same size. No one is really hooked.

I then pose the following problem: Cara wants to eat her candy, and since she sits near me, she has to break it into small pieces so I will not notice, so she divides her rectangle into sixteenths. Pauline hardly ever breaks the rules and decides she is too nervous to con-
continue after she breaks her candy bar into halves. Maria divides hers into quarters. I decide that I want a candy bar, too. I divide mine into thirty-seconds, smaller than Cara's pieces.

We decide that Cara eats seven pieces, Pauline eats one, and Maria eats three. I decide that I am just as rebellious as Cara, and I eat fourteen. They comment on my diet. On a large piece of chart paper, we make a drawing and put Xs on the pieces consumed. The girls actually show some animation now. We name the fractional part that each has eaten and label the drawing, which looks like this:

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The mood is more relaxed. I am confident that some fraction ideas, not just language, are becoming clearer. As we look at the drawing, I ask, "So, who ate the most?" All three look at me as if I am very silly, as if my question is so obvious as to be ridiculous. I ask again, "So really, who ate the most?"

*Cara:* You did, of course! [The other girls nod in agreement.]

*Teacher:* How do you know that?

*Maria:* You ate the most pieces.

*Cara:* You ate way more than anybody else. Pauline only had one piece, and Maria only had three.

*Teacher:* What does the drawing help you know about who ate the candy?

I am thinking that this is a pretty good question because I know they will "see" the answer in the representation. But Pauline answers, "What we said—Cara ate seven, I had one, Maria had three, and you had fourteen. You ate the most." What is it about the number of pieces and the relationship of that number to the whole that is not clear to them? What is it about the numerator that captures them and causes them to ignore the strong visual model? Is nothing about this problem familiar to them? Do we abandon this activ-
ity, or do I find a way to push it further? I try, “If I disagreed with you, how could you prove that you are right?” Nothing. Moments like this are so challenging.

I am thinking that, even with the drawing, they are not yet visualizing the parts eaten, the parts remaining, and how these parts relate to the whole, so I suggest that we actually each cut out our candy bars, then cut them into two parts—what we ate and what was left. We each put down the part we ate, and I ask again, “Who ate the most?” Three light bulbs come on, three students see something, and from the energy of it, I would guess it was for the first time. They observe that Maria ate almost the whole candy bar: “3/4, 75 percent,” they say. And they note that Pauline ate “1/2, 50 percent.” Clearly they have picked up something more than vocabulary during these last couple of weeks. They notice that sneaky Cara and the off-the-diet teacher ate the same amount. They appear on the edge of understanding something:

_Cara_: How come the numbers are different if the amounts are the same?
_Pauline_: I think it's like 3/6 and 4/8 and 5/10 are all 1/2. Different names for the same-size piece. [Pauline projects confidence as she says this—it sounds like a lot more than just words.]
_Cara_: But she ate more pieces than me.
_Maria_: Yeah, but she ate the same amount. You both have the same gone and the same left.

One of the difficulties in problematizing the curriculum for students who lack confidence and understanding in mathematics is that the teacher must consider both emotional and cognitive aspects of the students’ learning. The teacher must not only find or create the right problem but also bring students to a place where they are willing to enter the problem. In designing a problem that both provides the teacher with some insights into what the girls understand and gives them a way to clarify and move forward in their own mathematical understanding, Margery uses multiple sources of information. First she draws on her knowledge of the fundamental mathematical ideas in comparing fractions (as in Lucy’s class, the importance of the relationship of the fraction to the unit whole). Second, she uses her knowledge of problems and representations that have been effective in helping students compare fractions. For example, she mentions the loaf-of-bread problem that she has used to help students visualize how to compare fractional parts. Third, she must work with what she learns
from her real-time interaction with the students as she struggles to find points of entry and understanding.

Teaching students with a range of experiences, needs, competence, and confidence is not the exceptional situation; it is the situation that every elementary school classroom teacher faces. Although problems for students like these may, in comparison with what their peers are working on, seem simple, the tasks must be problematic for these students to engage them in working through mathematical ideas.

**Episode 4, “The Elevator Way”: Developing Representations and Metaphors**

As we saw in Margery's episode, representations of mathematical situations are important anchors for problematizing. Without the drawing of "candy bars," the three fifth graders would have had no way to compare the fractional parts. Looking only at the numbers, they focused on the numerators and compared the quantities as if the numerators alone indicated relative size. By representing the relationship of whole and parts and actually cutting out the parts they needed to compare, they were able to see the quantity represented by each fraction and to relate the numbers to the quantities (for example, seeing how two different fractions can indicate the same amount).

Representations and metaphors provide an anchor for students as they investigate problems and a basis for interaction with one another about their problem-solving process. Part of the teacher's role in problematizing the curriculum is to choose, develop, and help students develop representations and metaphors, then analyze whether and how these representations support students in their mathematical problem solving.

Curriculum materials and mathematics manipulatives offer many kinds of representations, some of which have been used for decades in mathematics education—for example, geometric solids, rectangular arrays, fraction pieces, and connecting cubes. But representations are not magic bullets. In Margery's episode, representing the fractional relationships with a picture did not automatically illuminate the mathematical ideas for the students. Margery had to work to engage the students with the representation in a way that helped them compare fractional parts. In the next episode, Sarah is listening to the ideas of her first- and second-grade students about subtraction. They have commonly used a number line and connecting cubes—two different ways to model subtraction. However,
class discussion reveals that students have different ideas about how to use these materials to model subtraction. One student comes up with a metaphor using a familiar situation that helps the students visualize the relationships on a number line.

For the past few years, I have been interested to note how my students come to make sense of the number line. It requires a certain amount of sophistication in their thinking, and although I encourage my students to use and make sense of a variety of different strategies, the number line presents a challenge that few of them seem to be able to resist trying out. The number line actually measures distance or spaces between numbers rather than concrete sets or objects, a concept that has seemed difficult for my students to fully understand at this age.

In my combined first-second grade this year, we have not discussed how to use the number line as frequently or in as much depth as in the past. Most of my second graders seem to have a fairly solid understanding of the concepts behind the number line, but my first graders are still struggling. I have also had the uncomfortable feeling that, because I haven’t given them much opportunity and time to discuss and explore their understanding of this strategy together, most of the first graders have gleaned “rules” for using the strategy, primarily from me.

Recently, as part of our daily routine, “Making Today’s Number,” many of my first graders and several of my second graders gathered to share “ways to make 12,” the day’s date. Many of the students had explained their answers and strategies for solving the problem. Latasha offered her solution, $18 - 6 = 12$, and explained that she used the number line to figure it out: “I counted back from 18, 17—that’s 1, 16—that’s 2, 15—that’s 3, 14—that’s 4, 13—that’s 5 and 12, that’s 6. So 18 minus 6 makes 12.” I accepted her articulate explanation and did not probe any further to see how clearly she actually understood what she was doing with the number line.

Will got up to share his answer and his strategy: “I came up with 18 minus 7 equals 12. I used the number line, and I counted down.” He demonstrated on the number line, pointing as he counted from 1 to 7, starting with the 18 and landing on 12:

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Keith commented, “I don’t think that works, cause I know 8 minus 7 equals 1, and that would make 11.” I assumed that Keith was think-
ing about subtracting 7 from the 8 in 18. Will looked confused at this point, so I intervened, "It looks like you and Latasha used the same strategy, and you both even started with 18, but you subtracted different numbers. Could this work? Let's try to figure it out."

Latasha explained her strategy again and then added, "Just like you told us, you never start on the number, you start counting down on the next number." I cringed. It sounded like my own voice regurgitated. How much of this did Latasha actually understand? I asked, "Why do you need to do that? Why not just start on 18?" Latasha replied, "Because you can't start on that number. You're supposed to start on the next one down." She shrugged.

I asked Will to explain his thinking a little bit more. He said, "Well, I wanted to take away from 18, cause I knew I had to get to 12 and that would be a takeaway, so I counted back." He demonstrated his strategy again. I asked, "Can anyone else help us out here? Both these strategies seem to make some sense, and also seem similar, but Latasha and Will thought about the problem differently. What should we be thinking about when we use the number line?"

Several students offered analogies using cubes, and I was wondering how thinking about counting cubes would help with the number-line model, in which the spaces between the numbers are counted, not a set of objects. I tried a different question, "Latasha started by counting down to 17 first instead of starting on 18. Does this work?" Many of my first graders looked confused at this point, but my second graders came to the rescue:

**Toshi:** Well, it's like you're already on 18, so you can't count 18 again. You have to count down.

**Teacher:** Can anybody explain what Toshi just said?

**Flavia:** You already counted 18. You are already there, so you can't count it again. So if you are taking away, you have to take away by going to the next number.

**Thomas:** I think I understand! Say I was at the Prudential Center. [Thomas loves buildings and landmarks.] If I was on the 18th floor and I wanted to go down on the elevator, if I wanted to go to the 12th floor, if I got on, I am at the 18th floor, so the next floor I go to is the 17th floor.

It took me a moment to realize how well this analogy worked. The class suddenly perked up, vitalized by concrete terms and images that they could visualize and understand. A buzz of conversation was heard as they discussed Thomas's idea. Will, whose incorrect
answer had catalyzed this conversation, said, “Yeah, I get it. If you
got on at the 18th floor and you go to 18, you don’t go anywhere.”
Keith acted Thomas’s idea out with his hands as he explained, “You
have to go to the next floor. The 17th floor is 1, 16th is 2, 15th—3,
14th—4, 13th—5, 12th—6! That’s 6 floors down!

This episode brought up several issues for me as a teacher. I would
like to continue to develop my ability to question students—push-
ing them toward higher-level thinking and clarifying their understand-
ing of concepts but without leading them to a prescribed solution. Latasha’s explanation of her problem was, in reality, a
recitation of a memorized rule I had given her, not a demonstration
of her understanding of a difficult concept. Latasha had given me
the answer I wanted to hear, and I mistakenly moved on without
probing any deeper.

However, I was very pleased with Thomas’s ability to synthesize
information and clarify a difficult concept and strategy with a very
simple, concrete example. I was also happy to find a larger number
of first graders using the number line successfully in our next few
sessions. They described this strategy as “the elevator way” of solv-
ing a problem. I am reminded of the importance of providing stu-
dents with the opportunity and meaningful context to think about
problem solving naturally through their own discussion. I feel as
though I am constantly struggling to find a middle ground—to
provide students with the opportunity to share their thinking and
explore a variety of strategies without losing many of my students.
They have difficulty listening to one another for long, but the sharing
component is a vitally important step in the development of
their thinking. This time I chose to push the discussion, and ulti-
mately, many of the students became engaged and gained a
stronger understanding of a difficult concept.

Sarah, like Lucy in the first episode, could simply have cor-
rected Will’s wrong answer. By asking her students to consider the
two solutions to the problem, she problematized the situation even
for students who had initially solved the problem correctly. Sarah
recognized that “correct” use of a representation can indicate that
students have learned a procedure only but have not constructed
for themselves how the representation models mathematical rela-
tionships. Sarah challenged her students to think through why
and how the number line can be used to represent subtraction.
Their discussion resulted in the development of a new metaphor
for understanding the “counting backward” method of solving
subtraction problems—which, in turn, offered all the students a
new way to think about this class of problems. Encountering, using, contrasting, and analyzing a variety of representations is one way of deepening the work that students are doing as they focus on core mathematical ideas.

Conclusion

No set of curriculum materials, no matter how carefully researched and tested, can anticipate all the ways in which ideas develop in a particular classroom’s mathematical community. In the four classroom episodes recounted here, the teachers all use curriculum materials to guide their work, but none of them expect the curriculum to do their teaching for them. They are actively engaged in learning from both the written curriculum and their students. In none of these cases did the teacher simply move on in the face of a confusing or difficult moment or try to settle an important mathematical question quickly. Their focus is not on how to “cover” the sequence of activities suggested by the curriculum but on how to engage students in thinking about significant mathematical ideas through judicious use of the written curriculum. They take seriously what the curriculum materials have to offer, but they are vigilant and analytic as students of their students’ thinking. They think hard about when to open the book for guidance—and when to close it so they can pay attention to what their students have to say.
Teacher Story 4

How Many 4s in 100?
Where Do We Go from Here?

Nancy Buell, fourth-grade teacher

In this episode, Nancy Buell faces two issues that all teachers face every day: (1) how to engage all students in the class in working on significant mathematical ideas, and (2) how to engage all students in class discussion about their ideas. These two intertwined challenges are inevitable as teachers consider the diverse needs and strengths represented in the classroom. In some situations, a single problem can engage all students at their own levels. At other times, a teacher must modify a problem slightly—perhaps by changing the numbers in the problem—to make it more accessible or more challenging. But sometimes the design of problems requires more differentiation to engage very different students in the same set of mathematical ideas. This kind of modification requires that teachers clearly identify the mathematics content students need to work on and assess the understanding that different students have about that content. In their study of factors of the multiples of 100, some students in this class have a sense of what a factor is and how it relates to a multiple. However, a significant number of students in the class do not have a well-developed understanding of the relationship between a factor and a multiple. As Nancy works with them, she realizes that they need to visualize and analyze how a factor relates to a multiple before they can profitably develop and test conjectures about the multiples of 100. The evolution of work over several days in her classroom illustrates her modification and differentiation of problems in response to students’ evolving understanding and the range of students’ needs (see Russell et al. in chapter 6 for further discussion of this theme).

—Susan Jo Russell
This week we worked from our curriculum materials to review the factors of 100 as well as the factors of some other numbers. In the discussion that followed, some students made observations and predictions about how knowing the factors of 100 would help them find factors of 400. Nell hypothesized that all the factors of 100 would be factors of 400. Pete speculated that 400 would have 4 times as many factors as 100 for a total of 36 factors; other students thought that it would not have that many. Nell and Gayle were fairly sure it would have no more than twice as many. So, of course, I set my plans aside, and the students worked to find all the factors of 400. They worked alone or in twos or threes. That day, I let them choose whomever they wanted to work with. Mostly they chose to work with students at their own level of understanding.

I noticed that about half the class found a lot of factors right away and soon were fairly sure they had identified them all. I met with the half that had finished, and we talked briefly about the strategies they had used to find the factors. Their strategies included the following:

- testing all of the factors of 100 to see whether they were factors of 400;
- testing multiples of 4, since 400 is 4 times 100;
- dividing by a known factor to find its factor pair; and
- looking at the factor pairs of 100 and multiplying one number of the pair by 4.

No one suggested trying every number in order or trying numbers at random. Most of the discussion in this group centered on predictions about factors of other multiples of 100. In general, they thought that the larger the multiple of 100, the more factors it would have. They were happy to go off and explore other multiples of 100 to test this conjecture.

I then met with the other half of the class. Most of this group had identified only a few of the factors of 400 at this point. We began sharing strategies, but the discussion was very different from my discussion with the first group. These students did not seem to have a plan for choosing the numbers they tested other than the factors of 100, an idea that had come up in our whole-group discussion before students began exploring on their own. This group tested numbers by skip counting. If they landed exactly on 400, they knew that the number by which they were skip counting was a factor. I sensed that their understanding of factors was limited to numbers that worked in this activity. Only
one student connected the idea with making rectangles using connecting cubes.

We all started this investigation together three days ago, and now we are in very different places! I am glad that I met with the two groups separately, because if we had had a whole-class discussion, it would have been dominated by the observations and predictions of the stronger group. I might well have missed the fact that a substantial number of students are still unsure of what a factor is and how it relates to multiplication and division.

Over the next two days, one group explored factors of other multiples of 100. I had asked the group to find the factors of other hundreds numbers, to look for patterns, and to think about how they knew whether they had found all the factors. I got together with them a couple of times to share their progress. They spontaneously pointed out patterns, made conjectures, and wondered aloud. They were interested in hearing and thinking about others' ideas. The discussion was lively, and the ideas just kept coming. Students said they were surprised that 400 had fewer factors than 300. They had thought, after finding factors for 100, 200, and 300, that the number of factors would keep increasing as the hundreds multiple increased. Someone mentioned that 100 and 400 had an odd number of factors, whereas 200 and 300 had an even number of factors. I asked why that outcome occurred and what they would predict about larger multiples of 100. They were off on a new exploration.

Meanwhile, I had posed a different problem to the second group, designed to focus their attention on how the factors of 100 were related to the multiples of 100. I asked them to find how many 4s are in 100, in 200, in 300, and so forth. Then they had to tell how they knew how many 4s are in 300 and also describe any patterns they noticed. They continued exploring how many 5s, 10s, 20s, or 25s are in the multiples of 100 up to 1000. Most of the students chose to work relatively independently, seldom checking in with one another. Very quickly, most of them abandoned other strategies and began dividing on the calculator to find the answers.

Although students were finding the correct answers, I was still dissatisfied. I believed that they had arrived at a method to get the right answer but that they had not necessarily acquired much understanding of why that answer was correct. They were not visualizing and analyzing how the multiplication relationship between 4 and 100 gave them information about the multiplication relationship between 4 and 200 or 300. I then posed a prob-
lem they had not done before, one that did not involve a factor of 100: I asked them what tools they might use to find how many 15s are in 300. They suggested using a calculator, a 300 chart (similar to a 100 chart but with thirty rows, with ten numbers in each row), money, and connecting cubes. I then asked the students to use the 300 chart, money, and cubes to model the relationship between the factor 15 and its multiple 300 to find the number of 15s in 300.

First the students worked as a whole group to skip count on the 300 chart by 15s. We marked an X on each count. When they got to 300, I asked where the answer was. They said it was the number of Xs we had made. Someone pointed out that you could count down the Xs in the fives column and multiply by 2 to figure out how many Xs appeared altogether, because for each X in the fives column, a corresponding X appeared in the tens column.

Working with money to find the number of 15s was not obvious, since we do not have a 15-cent coin, but Angela made piles of one dime and one nickel to build up to one dollar. She ended with one dime left over. Then she made identical arrangements for the other two dollars. She counted the piles and got eighteen 15s. She said that the three leftover dimes were 30 cents, or two more 15s, for a total of 20 fifteens.

The students seemed engaged, with everyone looking and listening. Most of them offered comments along the way—in contrast with the way most of them operate in whole-class discussions. The pace of the discussion was slower, and the students had time to figure things out. We were able to stay focused on the question of how to use the tools and how each number was represented.

Before I asked anyone to show how he or she might use the cubes, I asked the students to think about it. Then I had them verbally share their ideas with a partner. I wanted to make sure that no one was going to let someone else do the thinking, since we would have only enough time for one person to share a strategy. Gary said that he could do it by making groups of 30 cubes. Laurel tried to correct him, but I suggested that we should see what he meant. Carefully counting to make sure he had 10 cubes on each stick, Gary made groups of 3 sticks plus 1 extra stick to make 100. He did the same for two more hundreds. Then he made the extras into another group of 30. He then counted the groups of 30 by twos to find the number of 15s. I was amazed that Gary—who is my weakest student in mathematics and who still has to count the sticks to be sure that each contains ten cubes—could come up with such a strategy. At this point the students were
already late for art class. As Nina said, "When math is fun, it's easy to forget about time." Of course, that comment made my day!

I think that both groups of students benefit from working separately at times on problems designed for them. In whole-class discussion, we sometimes get off on ideas that intrigue some students while others have not finished grappling with foundational ideas and strategies. With both groups working on problems that engaged them in thinking—at their level of understanding—about the relationships of factors and multiples, both groups were able to proceed at a comfortable pace, present their ideas with confidence, and get excited about exploring mathematical ideas.