Conic Sections
Expository Paper

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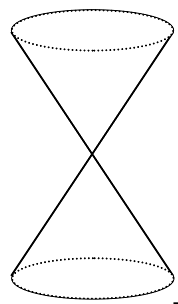
Research: Conic Sections

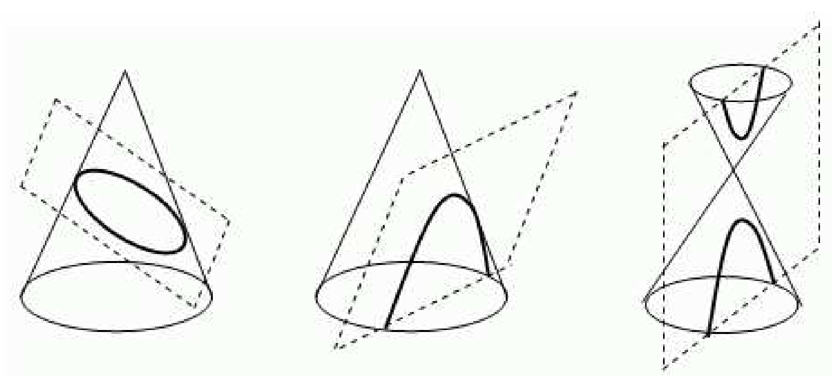
Conic sections, or conics, include the various geometric figures created by the intersection of a plane with a cone. It is important to note that the definition of a cone includes the surface generated by a straight line that moves so that it always intersects the circumference of a given circle and passes through a given point not on the plane of the circle. The point, called the vertex of the cone, divides the cone into two “halves” called *nappes*.



Practically speaking, these two nappes look, to the lay viewer, like two separate cones set end on end (see figure 1). This figure is sometimes referred to as a double cone.

Figure 1

The examples in this paper are created from right circular cones (cones whose vertex is directly below the center of a cross sectional circle) like the cone shown in Figure 1. There are three curves called conic sections (see figure 2), each unique in its construction and definition. These include the parabola, the ellipse, and the hyperbola. In some cases, the circle is listed as a fourth conic. To be precise, though, a circle is really a unique ellipse.



ellipse parabola hyperbola

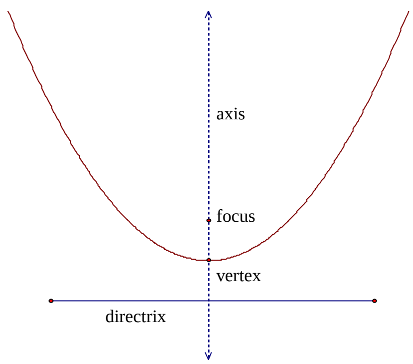
Figure 2

Conic sections were first extensively studied and written about by Euclid and later Apollonius of Perga more than 2000 years ago. These early mathematicians studied conics in the context of their geometric properties. Descartes, in his work *Geometry*, later studied these curves algebraically.

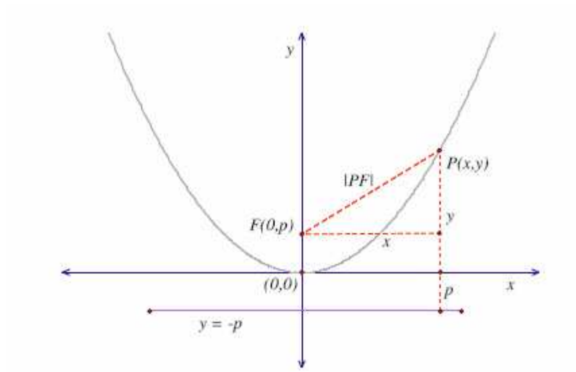
The curves that comprise the conic sections are interesting both mathematically and scientifically. Kepler discovered conics in the orbits of planets about the sun. We can see conics in suspension bridges, satellite dishes, automobile headlights, and telescopes. Physicians apply conics in treatment of kidney stones. John Quincy Adams even took advantage of conic curves to eaves drop on members of the House of Representatives from his desk in Statuary Hall in the U.S. Capitol building. Indeed, these curves are profoundly relevant to the world in which we live.

The Parabola

The parabola is a one-piece non-closed curve formed by a plane that intersects only one nappe of the cone. This plane is parallel to exactly one element of the cone (e.g. one of the lines connecting the vertex of the cone to the circumference of its base). Described in terms of the plane, a parabola is defined as the set of points in a plane that are equidistant from a fixed point F, called the focus, and a fixed line called the directrix. The vertex of a parabola is the point on the curve half way between the focus and the directrix. The axis is a line through the focus and perpendicular to the directrix. The vertex is also positioned on the axis.



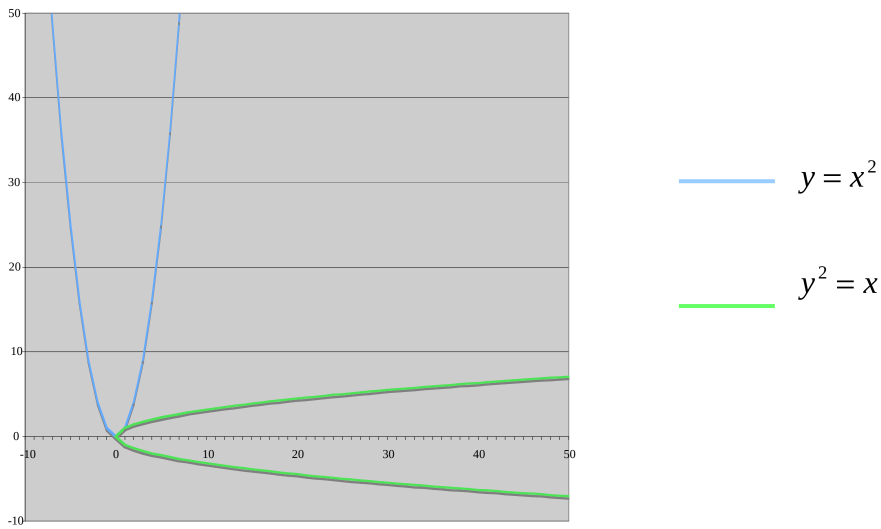
The simplest equation for a parabola can be found by placing its vertex at the origin $(0,0)$ and its directrix parallel to the x-axis. If the focus is at point $(0,p)$, we can say that the directrix is the line $y=-p$, since we know that the focus and directrix are equidistant from the vertex. We also know that any point $P(x,y)$ on the parabola is, by definition, equidistant from the focus and the directrix. This knowledge allows for the construction of an equation.



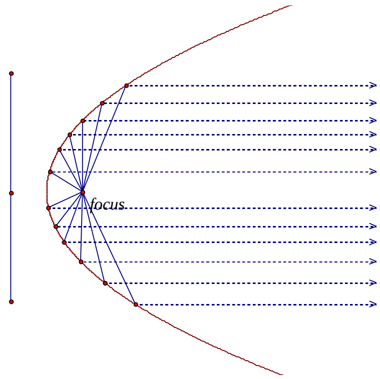
As shown in the figure above, the distance from a given point $P(x,y)$ to the focus $F(0,p)$ can be seen as the hypotenuse of a right triangle. The base (one leg) of the triangle has a length of x, and the second leg of the triangle has a distance of $|y-p|$. Using the Pythagorean theorem, we find $|PF|=\sqrt{x^2+(y-p)^2}$. The distance from P to the directrix is $|y+p|$. As stated, the definition of parabola says these two distances are equal. This gives us $|y+p|=\sqrt{x^2+(y-p)^2}$. This equation can be simplified:

$$\sqrt{x^2+(y-p)^2}=|y+p|$$
$$x^2+(y-p)^2=(y+p)^2$$
$$x^2+y^2-2py+p^2=y^2+2py+p^2$$
$$x^2=4py$$

With this equation, a parabola opens upward if $p>0$ and downward if $p<0$. We can also find the parabola $y^2=4px$ that opens to the right if $p>0$ and to the left if $p<0$. Two simple algebraic expressions whose graphs are parabolas are shown on the graph below:

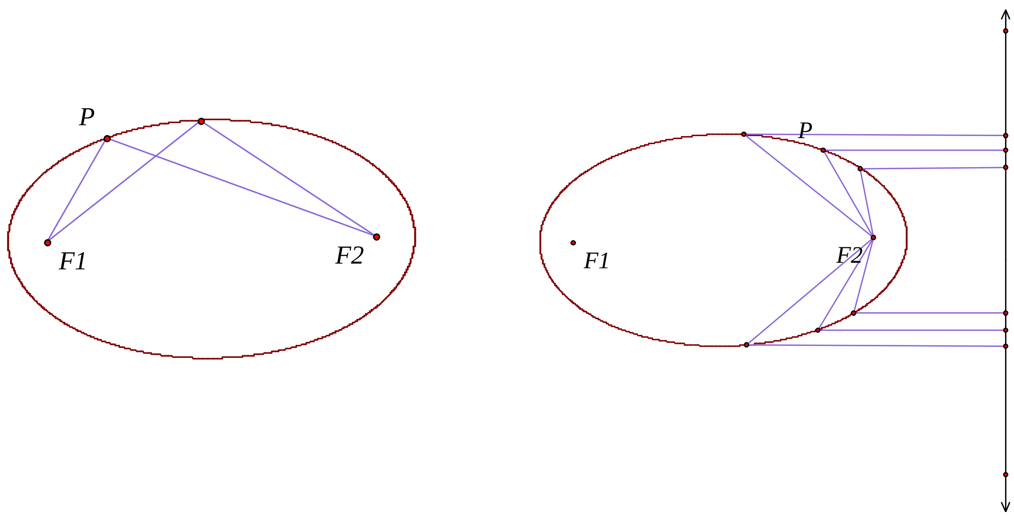


According to Galileo, the path of a projectile shot into the air at an angle to the ground is a parabola. Parabolas have many other interesting applications as well. The cables on suspension bridges form parabolic curves. Telescopes and automobile headlights also rely on parabolic curves. Parabolas have a reflective property whereby any light source placed at the focus will reflect at any given point on the curve along a line parallel to the x-axis. This trait allows one to aim reflected beams of light in a particular direction as shown below:

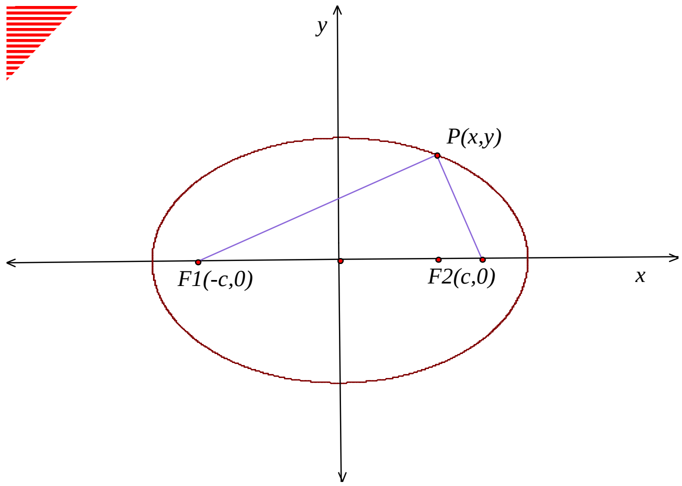


The Ellipse

The ellipse is a closed curve formed by a plane that intersects only one nappe of a cone in such a way that the plane also intersects all lines that connect the vertex of the cone to the circumference of the base of the cone. Described in terms of the plane, an ellipse is defined as the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. Points F_1 and F_2 are called the foci. The ellipse can also be defined as the set of all points whose distance from a single focus is proportional to the horizontal distance from a directrix.



The simplest equation for an ellipse is derived in a similar fashion to the equation for a parabola. By placing the foci on the x-axis at points equidistant from the origin $(0,0)$, we are able to again find right triangles with which we can determine the distance to any point $P(x,y)$ on the ellipse. Suppose the foci have a distance of c from the origin $(0,0)$ so that they are located at $F_1(-c,0)$ and $F_2(c,0)$.



As shown in the figure above, the distance from point $P(x,y)$ to focus $F_1(-c,0)$ forms the hypotenuse of a right triangle with one leg whose length is equal to y and another leg whose length is $(x+c)$. We can use the Pythagorean theorem to determine the distance $|PF_1| = \sqrt{(x+c)^2 + y^2}$. Likewise, the distance from point $P(x,y)$ to focus $F_2(c,0)$ forms the hypotenuse of another right triangle. Using the same technique, we can find the distance $|PF_2| = \sqrt{(x-c)^2 + y^2}$. If we let the sum of these distances $|PF_1| + |PF_2|$ be $2a$, we can simplify to find the equation for an ellipse:

$$|PF_1| + |PF_2| = 2a$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-4cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2}$$

$$-cx = a^2 - a\sqrt{(x+c)^2 + y^2}$$

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

$$a^2(x+c)^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$a^2(x^2 + 2xc + c^2) + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2$$

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

If we then let $b^2 = a^2 - c^2$, we derive the equation $b^2x^2 + a^2y^2 = a^2b^2$. We can divide both sides by a^2b^2:

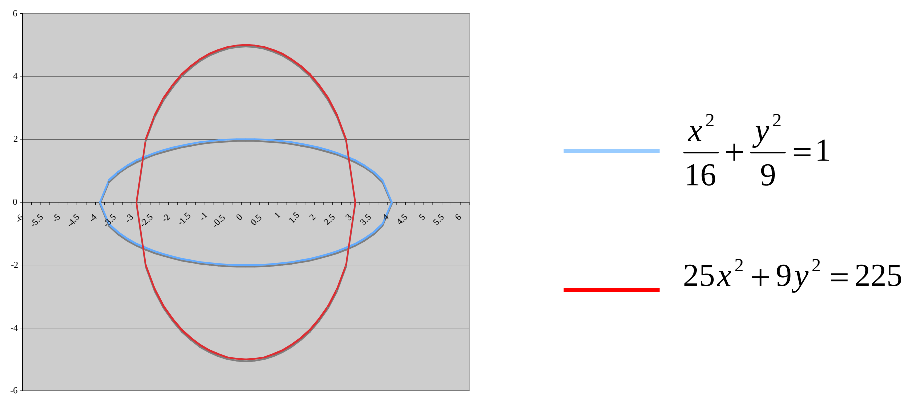
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This ellipse has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, on the x-axis. The points at which the ellipse intersects the y-axis $(0, \pm b)$ are called vertices. The ellipse intercepts the x-axis at $(\pm a, 0)$. The greater of the two (a or b) denotes the "major axis", while the other marks the "minor axis".

We can also interchange x and y in the equation to show an ellipse with foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, on the y-axis. The vertices of this ellipse are located at $(0, \pm a)$. The equation is as follows:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Two simple algebraic expressions whose graphs are ellipses are shown in the graph below:

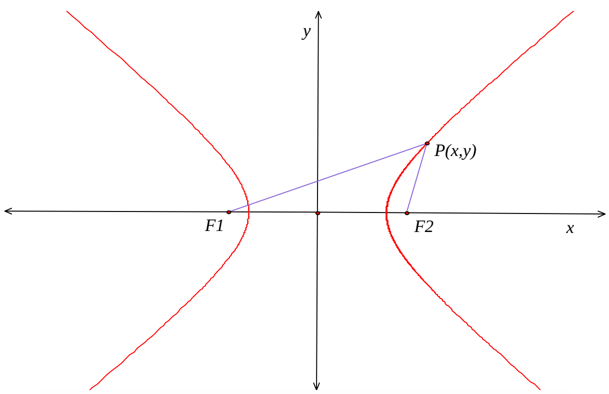


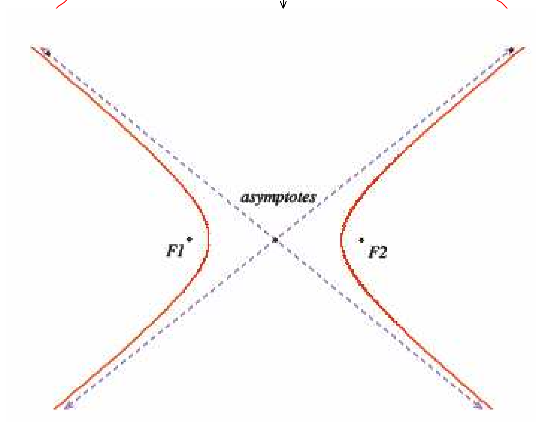
Like the parabola, the ellipse is an interesting and useful curve. Though according to Plato, only circular motion about the earth was possible in the heavens, the planets themselves just didn't behave accordingly. Copernicus later suggested that the sun might be at the center of planetary motion. Around 1600, Johannes Kepler found that the motion of the planets was, in fact, elliptical with the sun as one focus – an idea later proved by Isaac Newton.

Ancient planetary science is not the only application of elliptical curves, though. Ellipses also have a reflective property such that if a light source is placed at one focus, all light reflected off the surface of the ellipse will be reflected **onto** the other focus. This trait of elliptical curves is used today in the treatment of kidney stones. In a process called lithotripsy, sound waves are emitted from a point arranged in such a way that together with a patient's kidney stone, the foci of an ellipse are formed. Focused bursts of sound waves travel harmlessly through the patient's body and are reflected onto the kidney stone, breaking it into small pieces so that it can pass easily out of the body.

The Hyperbola

The hyperbola is a two-piece non-closed curve formed by a plane that intersects both nappes of the cone. Described in terms of the plane, a hyperbola is defined as the set of points in a plane the difference of whose distances from two fixed points F_1 and F_2 is a constant. The hyperbola's two parts are called branches. As each branch is extended, the curve approaches lines called the asymptotes, as shown in the second figure below:



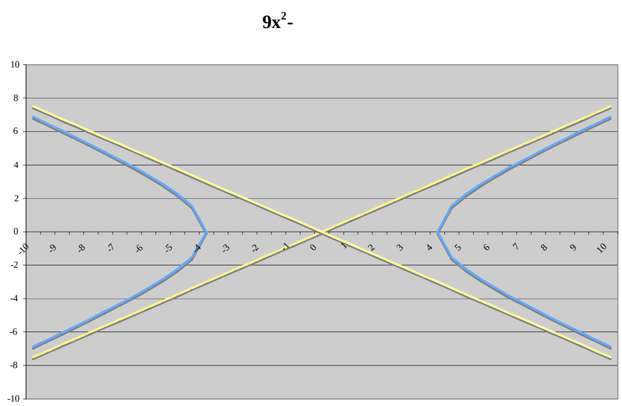


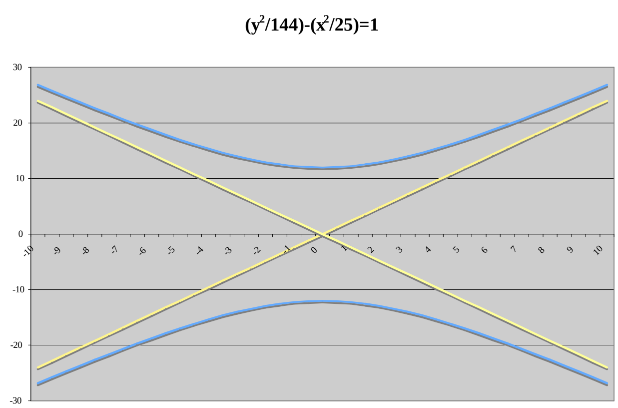
A simple algebraic formula for the hyperbola can be derived logically in the same manner as for the ellipse. Points on the ellipse are derived from the sum of distances $|PF_1|+|PF_2|$. In contrast, points on the hyperbola are derived from the **difference** of distances $|PF_1|-|PF_2|$. Using similar right triangle constructions and simplifying, the formulas for hyperbolas looks very similar to the formulas for ellipses:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad OR \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The first formula yields a hyperbola whose foci $(\pm c, 0)$ are on the x-axis, where $c^2 = a^2 + b^2$. The vertices are located at $(\pm a, 0)$, and asymptotes are $y = \pm(b/a)x$. The second formula yields a hyperbola whose foci $(0, \pm c)$ are on the y-axis, with vertices $(0, \pm a)$ and asymptotes $y = \pm(a/b)x$.

Two simple algebraic expressions whose graphs are hyperbolas are shown below:





The hyperbola has its own unique applications in our world. You might see a hyperbolic curve in the pattern cast on a wall by a cylindrical lampshade. Two stones cast into a pond create concentric circles. The intersection of these circles forms a hyperbolic curve. This same idea is used in RADAR tracking devices such as LORAN telescopic lenses. If three stations are tracking a signal, any two can be used to narrow the object’s location down to a specific hyperbolic curve. The intersection of the third station’s data can then pinpoint the object with certainty.

Eccentricity

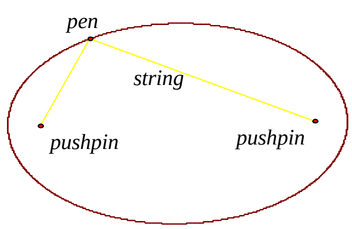
Although each of the conic sections can be defined as we have seen, there also exists a theorem of eccentricity that gives a single definition for any conic section. According to this theorem, the eccentricity e of a curve is the fixed positive ratio of the distance of any point $P(x,y)$ on the surface of the curve to the focus F to the distance of point $P(x,y)$ to the directrix l:

$$\frac{|PF|}{|Pl|} = e$$

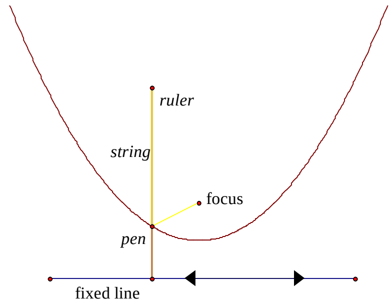
The conic is defined by the magnitude of this ratio. That is to say if $e < 1$, the curve is an ellipse. If $e = 1$, the curve is a parabola. If $e > 1$, the curve is a hyperbola.

Physical Models

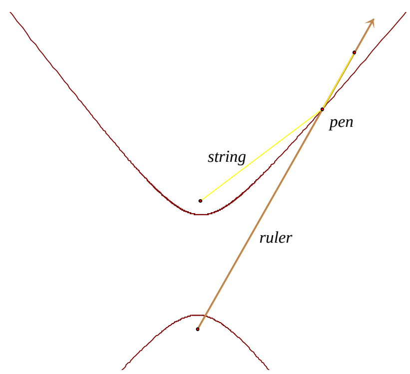
The conics can also be generally represented with physical constructions. Perhaps the easiest to construct is the ellipse. Place two pushpins into a surface and loop a length of string about them. Use a pen to extend the loop of string, tracing about the perimeter of the ellipse.



Although slightly more complicated, the parabola can be created with a similar string and pen technique by moving a ruler (with a groove cut at the center along its length) back and forth along a fixed line, keeping the ruler perpendicular to the line. In this case, a fixed length of string will connect at one end to a focus point, and at the other to the end of the ruler. The pen should be held in the ruler so that the string is kept taut.



The method for creating a hyperbola is very much like creating a parabola. In this case, however, one end of the ruler is fixed at one focus point, while the other end is moved in an arc. The fixed length of string connects the free end of the ruler to the remaining focus point. Again, the pen is arranged to hold the string taut while the ruler moves, tracing one branch of the hyperbola. The same process can be repeated, trading focus points, to create the other branch.



Equations and Sample Problems

Every conic section can be written as an 2nd degree polynomial of two variables as follows:

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

If $B^2 < AC$, the equation will yield at ellipse, circle, point, or no curve

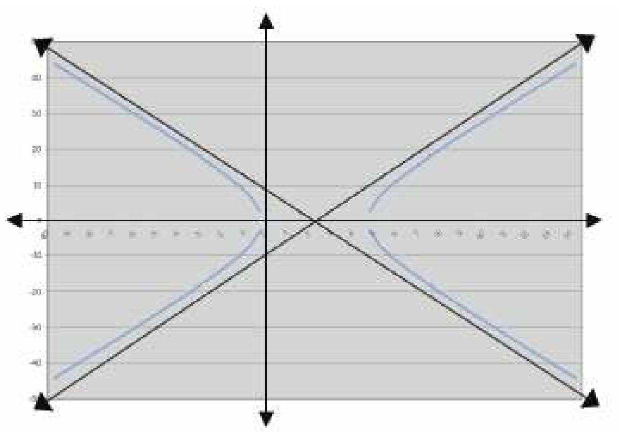
If $B^2 = AC$, the equation will yield a parabola, 2 parallel lines, 1 line, or no curve

If $B^2 > AC$, the equation will yield a hyperbola or 2 intersecting lines

Hyperbola:

$$2y^2 - 3x^2 - 4y + 12x + 8 = 0$$
$$2y^2 - 4y - 3x^2 + 12x = -8$$
$$-2(y^2 - 2y) + 3(x^2 - 4x) = 8$$
$$-2(y^2 - 2y + 1) + 3(x^2 - 4x + 4) = 8 - 2 + 12$$
$$\frac{-2(y-1)^2}{18} + \frac{3(x-2)^2}{18} = 1$$
$$\frac{(y-1)^2}{-9} + \frac{(x-2)^2}{6} = 1$$
$$\frac{(x-2)^2}{6} - \frac{(y-1)^2}{9} = 1$$
$$a^2 = 6$$
$$b^2 = 9$$
$$c^2 = 15$$

$$foci\ (2 \pm \sqrt{15}, 1)$$
$$vertices\ (2 \pm \sqrt{6}, 1)$$
$$asymptotes\ (y-1) = \pm\frac{\sqrt{6}}{3}(x-2)$$

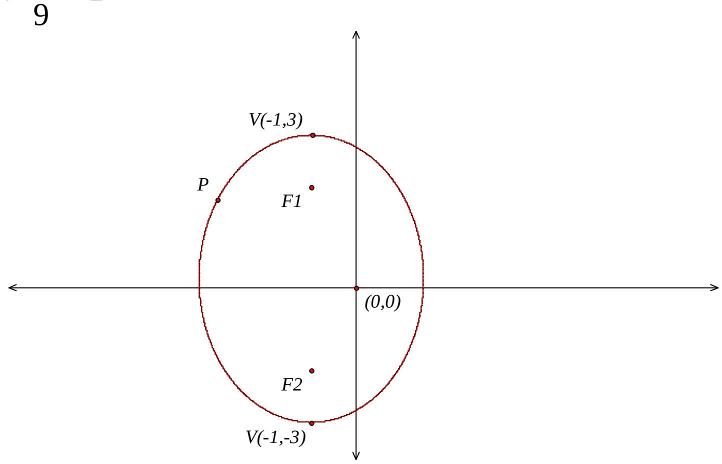


Ellipse:

$$9x^2 - 18x + 4y^2 = 27$$
$$9(x^2 - 2x) + 4y^2 = 27$$
$$9(x^2 - 2x + 1) + 4y^2 = 27 + 9$$
$$\frac{9(x-1)^2}{36} + \frac{4y^2}{36} = 1$$
$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$$

$$a^2 = 9$$
$$b^2 = 4$$
$$c^2 = 5$$

$$\textit{foci } (-1, \pm\sqrt{5})$$
$$\textit{vertices } (-1, \pm 3)$$

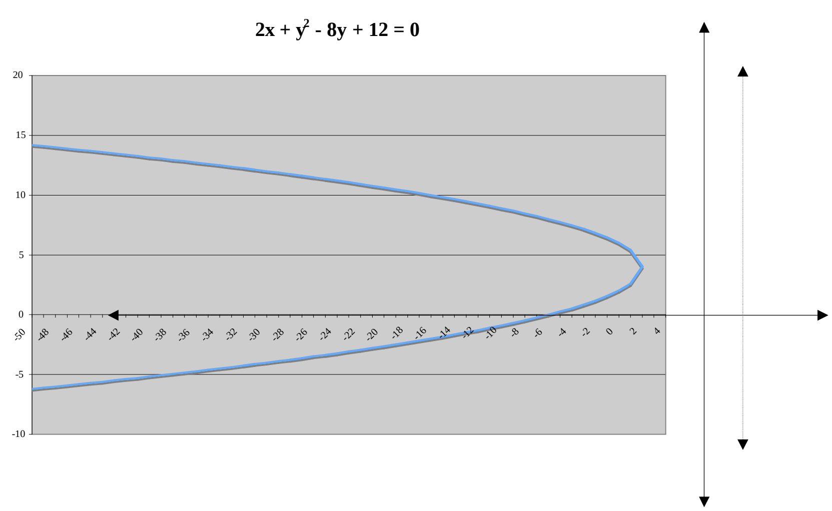


Parabola:

$$2x + y^2 - 8y + 12 = 0$$
$$y^2 - 8y = -2x - 12$$
$$y^2 - 8y + 16 = -2x - 12 + 16$$
$$(y-4)^2 = -2x + 4$$
$$(y-4)^2 = -2(x-2)$$

$$k = 4$$
$$h = 2$$

$$\textit{vertex } (2, 4)$$
$$\textit{focus} \left(\frac{3}{2}, 4\right)$$
$$\textit{directrix}: \quad x = \frac{5}{2}$$



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