

Symmetry of Scale

Expository Paper

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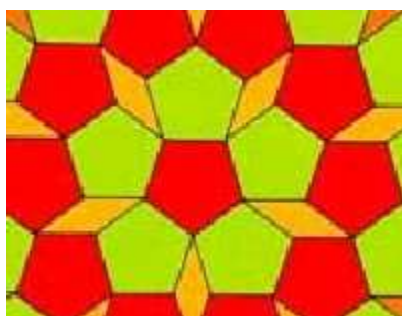
MAT Exam – Part IB

A checkerboard, a beehive, a brick wall and a mud flat dried in the sun all have something in common. They are all examples of tilings or tessellations. Although you may think of mosaics or other pieces of artwork when you hear these words, in actuality you should also think of mathematics and science. I will describe in more detail the mathematics involved with tessellations and tilings, and discuss specific tilings such as the Pinwheel Tiling and the Penrose Tiles.

Dr. Math, from the “Ask Dr. Math” website, defines the word “tessellate” as: “to form or arrange small pieces (like squares) in a checkered or mosaic pattern”. Dr. Math also explained that “the word ‘tessellate’ is derived from the Ionic version of the Greek word ‘tesseres,’ which in English means ‘four’” (Drexel University, 1994-2006).

Although these are helpful in understanding what tessellations are, these descriptions are quite limited. Tessellations are “created when a shape is repeated over and over again covering a plane without any gaps or overlaps” (Drexel University, 1994-2006). The definitions mentioned previously refer to squares which is the reason for the reference to the number four. However, tessellations can involve any shape. (See figure 1)

Figure 1:



Pentagons and Quadrilaterals



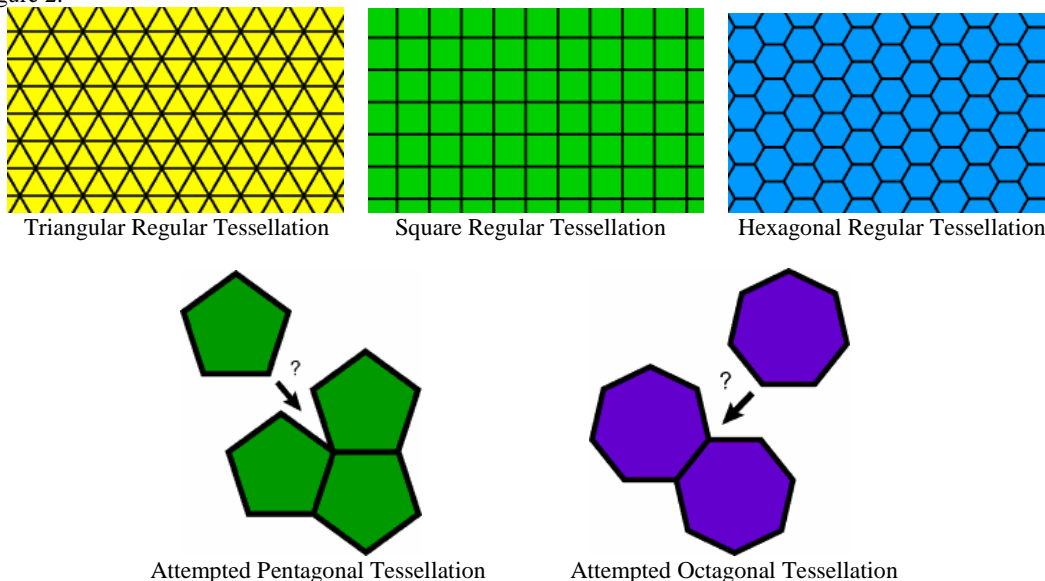
Tile from Taj Mahal

We often think of tessellations in the 2-dimensional Euclidean Plane. However, tessellations can also take place with 3-dimensional figures such as a sphere. An example of a 3-dimensional tessellation is a soccer ball. The study of tessellations dates back to 1619 with the study of Johannes Kepler. More research with tessellations took place in 1891 with the Russian crystallographer E.S. Fedorov. There has also been more recent research from the 1950's to the most recent "discovery" in 1994.

As you can see from the definition, tessellations can vary greatly as there are not many stipulations to create tessellations. There are three special types of tessellations, though. These are tessellations that involve only one type of regular polygon, thus creating the name regular tessellations. Regular polygons are polygons with all angles having the same measurement and all sides having the same lengths. Because of these qualities and the fact that the sum of the angles of any triangle total 180 degrees, one can find the degree measurement of all regular polygons. For example, a regular triangle has three 60-degree angles found by dividing 180 by the number of angles (3). A regular quadrilateral (a square) can be divided into two triangles, which means the angles sum 360 degrees. By division, we now know that each angle of a regular quadrilateral must be 90 degrees. This method of dividing polygons into triangles to find the angle measurements can be done for other polygons as well. In general, the formula for finding the measure of the angles of a regular polygon is $[(n-2) * 180]/n$ in which n = the number of sides of the polygon. One can find that regular pentagons have angles of 108 degrees; regular hexagons have angles of 120 degrees, etc. Each vertex of a regular tessellation must look the same and each vertex represents 360 degrees. Combining all of these facts will show that regular triangles, squares and hexagons are the figures with angles that are

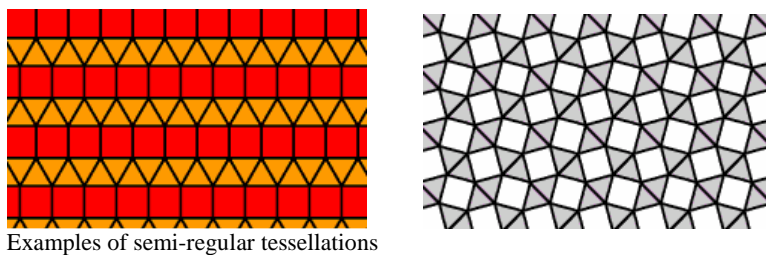
factors of 360, which means they can be put together to form regular tessellations. (See figure 2 for regular tessellations and attempted regular tessellations)

Figure 2:



Another type of tessellation is called semi-regular tessellations. These tessellations are made up of at least two different regular polygons. (See figure 3)

Figure 3:

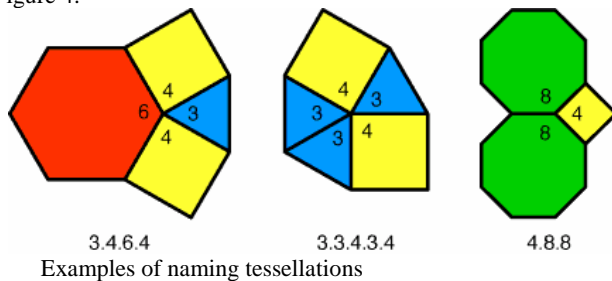


Examples of semi-regular tessellations

Tessellations are named by the vertex and the shapes that meet at that vertex. For example, if there are four squares that meet at a vertex, it would be named 4.4.4.4. This is because of the number of sides for each of the shapes that meet at the vertex. If there are four triangles and a hexagon that meet at a vertex, it would be named 3.3.3.3.6. The numbers listed in the name of the tessellation should be listed in chronological order.

(See figure 4)

Figure 4:



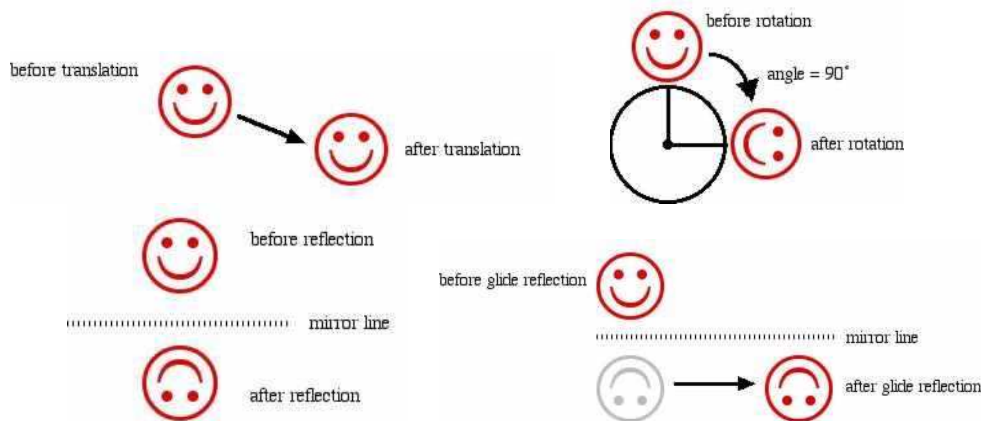
The words “tilings” and “tessellations” are often times used interchangeably. There are some differences, however. Tilings usually refer to straight-edged shapes - polygons. Tessellations are not as restrictive – allowing shapes with curved edges. Even with the restriction of polygon shapes, there are an infinite number of types of tilings. Tilings can be categorized by repeating or non-repeating patterns. Repeating patterns are called periodic tilings and the non-repeating patterns are called aperiodic tilings. The periodic tilings must repeat in a regular way. This means that the tiling (for the 2-dimensional plane) repeats in “two independent directions” (Geometry Technologies, 1999). A test can be done to determine if the tiling is periodic or not. This is called the lattice test. A lattice is a grid of horizontal and vertical parallel lines. A tiling is periodic when the lattice can be placed on the tiling so that the figures inside each section of the lattice is identical to the figures inside the other sections of the lattice. Tilings can also be categorized by the number of tile types used to create the tiling. If a tiling is created with just one type of tile, it is called a monohedral tiling. If it is created with two types of tiles, it is called a dihedral tiling.

Symmetry is another topic with tilings. Science U defines symmetry in this way: “A figure in the plane is symmetric if you could pick up a copy of it, move it around to a new location somehow, and set it back down on the original figure so that it exactly matches up again” (Geometry Technologies, 1999). People, in general, find beauty in

symmetrical objects. Symmetry gives the sense of balance and peacefulness. Judith Lanlois of the University of Texas at Austin researched people’s reactions to photographs of different faces. She found that the faces chosen as beautiful were the ones that were the most symmetrical. (Burger & Starbird, 2000, p. 250). There are two types of symmetries that can be found with patterns in the 2-dimensional Euclidean plane. They are: rigid symmetries and symmetries of scale. Rigid symmetry of a pattern in a plane “is a motion of the plane that preserves the pattern and does not shrink, stretch, or otherwise distort the plane” (Burger & Starbird, 2000, p. 251). Four kinds of symmetry that correspond to the ways in which a figure can be moved are also called isometries or rigid motions. The four isometries are: (See figure 5)

- * Translations: this refers to sliding a figure
- * Rotations: this refers to turn a figure about a fixed point
- * Reflections: this refers to flipping a figure over a line
- * Glide Reflections: this refers to a translation and a reflection

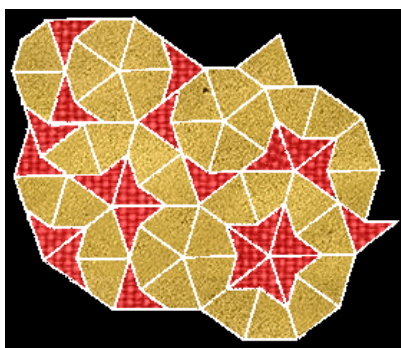
Figure 5:



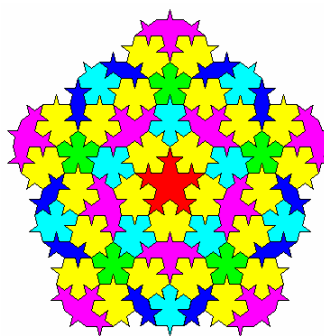
Two or more of these symmetries may be done to create a new symmetry. This is described as: “the composition of two isometries is again an isometry” (Geometry Technologies, 1999).

Symmetry of scale is the other type of symmetry that can be found with patterns in the plane. Symmetry of scale is also referred to as being scalable. Patterns are scalable if “the tiles that make up the pattern can be grouped into super-tiles that still cover the plane and, if scaled down, can be rigidly moved to coincide with the original pattern” (Burger & Starbird, 2000, p. 252). A pattern consisting of squares or a pattern consisting of equilateral triangles are examples of scalable patterns. For example, when looking at a pattern of squares, a grouping of 2×2 squares can be enlarged and still fit the pattern or likewise, the grouping of 2×2 squares can be made smaller and can still fit the original pattern. These examples (a pattern of squares or a pattern of equilateral triangles) show both symmetry of scale and rigid symmetry. There are some patterns, however, that they are scalable but do not have rigid symmetries. This topic is one of recent research and discovery. It was in the mid-1960’s that Robert Berger discovered examples of scalable patterns with no rigid symmetries. His patterns were very complicated and consisted of thousands of different types of tiles. In the 1970’s, Roger Penrose continued with this research and discovered patterns that are scalable and have no rigid symmetries with the use of only two types of tiles. These patterns are now named after Roger Penrose, called the Penrose Tiles. The two types of tiles Penrose used resemble kites and darts. Every tile used has been placed in one of ten possible orientations in the plane. (See figure 6)

Figure 6:



Examples of Penrose Tilings



Penrose discovered many other types of aperiodic (non-repeating) patterns. They all fit the rule that no two tiles can be touching to form a single parallelogram. Some unique characteristics of one of the patterns Penrose discovered are that they have rotational symmetry around the center point and have reflection symmetry about a line. The tiling, however, is aperiodic. This means that there is no translational symmetry; it never repeats exactly. There are also unique characteristics related to the golden number Phi. When the two types of tiles are placed together, there is a Phi: one or one: Phi ratio relationship. The science world has realized that the Penrose patterns are seen in nature as some atoms and crystals are arranged in this way.

Research of scalable patterns with no rigid symmetries continued into the 1990's when John Conway of Princeton University and Charles Radin from the University of Texas, Austin discovered the Pinwheel Pattern. The Pinwheel Tiling is a scalable pattern with no rigid symmetries using only one tile. That one tile, however, occurs in infinitely many orientations. The tiling is made from Pinwheel Triangles. The Pinwheel Triangle is a right triangle with one leg two times as long as the other. The Pythagorean Theorem can be used to show that the hypotenuse for these triangles is the square root of five. Five Pinwheel Triangles can be combined to form one larger Pinwheel Triangle, called a super-tile or a 5-unit Pinwheel Triangle. (See figure 7)

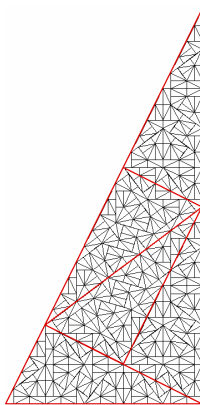
Figure 7:



Example of a 5-unit Pinwheel Triangle

This duplicating and enlargement can continue. Since there are no rigid symmetries, it means that the triangles can only be placed (or fit) into the pattern in just one way. When there is only one way to group tiles to form a super-tile, it is said to have Uniqueness of Scaling (Burger & Starbird, 2000, p. 258). This arrangement of the five triangles in the Pinwheel Tiling is called the T-arrangement since the “T” shape is formed when the triangles are in place. (See figure 8)

Figure 8:



Example of many Pinwheel Triangles

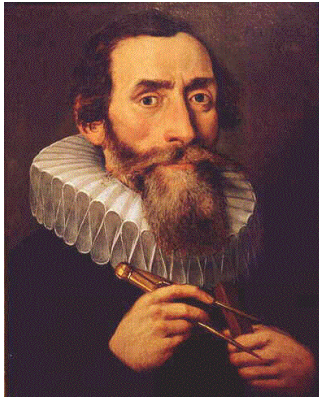
There have been many mathematicians that have contributed to the study and knowledge of tilings and tessellations. One of the first mathematicians in this area of study was Johannes Kepler. (See figure 9) Kepler lived from 1571 – 1630 in Weil der Stadt, Wurttemberg, Holy Roman Empire (which is now Germany). Kepler’s father was a soldier and died in war and his mother was the daughter of an innkeeper. After Kepler’s father died, Kepler and his mother lived in his grandfather’s inn. Kepler was a religious person, intending to be ordained. He studied at the University of Tubingen. After his first year there, Kepler earned all “A’s” except for mathematics. Kepler married and had children. Kepler’s seven year old son and wife died. Even during difficult times, Kepler remained positive and religious. Kepler later remarried. He is best known

for his work with planetary motion, optics and polyhedra (University of St. Andrews – Scotland, 1999).

Maurits Cornelius Escher lived in the Netherlands from 1898 – 1972. (See figure 9) His father, who was a civil engineer, and his second wife, who was the daughter of a government minister, raised Escher. Escher was said to have a mathematical mind, but he never excelled in mathematics. Instead, he preferred music and carpentry and was very creative. Escher was trying to fulfill his family's dream that he becomes an architect, but he failed too many classes and did not graduate. Escher found another school that allowed him to try the study of architecture, but his lack of motivation and poor health prevented him again from graduating. Escher traveled frequently and focused on art and sketching. Escher married and had three children. He continued to focus on art and found an interest with symmetry, which is what exposed him to mathematics. Escher is known for his symmetrical drawings (University of St. Andrew – Scotland, 1999).

Roger Penrose was born in 1931 in England. (See figure 9) Penrose's parents both worked in the medical field. Penrose's father later became a professor at University College London and Penrose attended school there for free. He studied mathematics and received awards for his achievement. He then attended Cambridge to study Pure Mathematics. Penrose began publishing articles and received numerous awards for his work with mathematics (University of St. Andrew – Scotland, 1999).

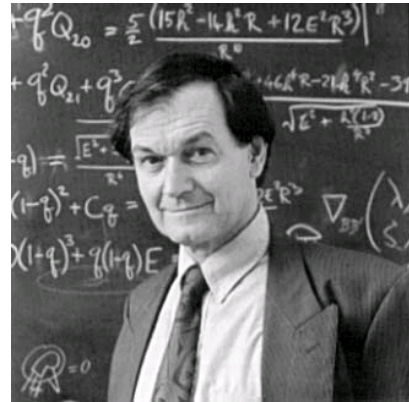
Figure 9:



Johannes Kepler



M.C. Escher



Roger Penrose

As you have seen, the study of tessellations and tilings has spanned hundreds of years. Tessellations are pattern that do not create gaps or overlappings. Tilings are a type of tessellation that only uses polygons. There are three regular tessellations – using triangles, squares and hexagons. There are two types of symmetry found with patterns in the 2-dimensional plane: rigid symmetry and symmetry of scale. The Penrose Tiles are an example of a pattern that has symmetry of scale but no rigid symmetry. The Pinwheel pattern is another example of a pattern that has symmetry of scale, no rigid symmetry, and uses only one type of tile. Several mathematicians have contributed to this topic that is complex, yet practical as all of us see these patterns in science and in nature as we look at beehives, mud flats dried in the sun, crystals, . . .

References

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