Just What Do You "Mean"?
Expository Paper

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In Ancient Greece the Pythagoreans were interested in three means. The means were the arithmetic, geometric, and harmonic. The arithmetic mean played an important role in the observations of Galileo. Along with the arithmetic mean, the geometric and harmonic mean (formerly known as the subcontrary mean) are said to be instrumental in the development of the musical scale. As we explore the three Pythagorean Means we will discover their unique qualities and mathematical uses for helping us solve problems.

Consider 2 positive real numbers, *a* and *b*, where $b \geq a$. (This will be true throughout the entire document unless noted otherwise).

<u>Arithmetic Mean:</u>

The Arithmetic Mean (A.M.) of two numbers is the quotient of the sum of the two numbers and the number two.

That is $\text{A.M.} = \dfrac{(a+b)}{2}$.

For example: Let $a = 4$ and $b = 9$ then the Arithmetic Mean would be $\text{A.M.} = \dfrac{(4+9)}{2} = \dfrac{13}{2} = 6.5$

<u>Geometric Mean</u>

The Geometric Mean (G.M.) of two numbers is the square root of the product of the two numbers. This can be written as a proportion.

Let x represent the geometric mean

$\dfrac{a}{x} = \dfrac{x}{b}$ and through the use of cross products the result is $x^2 = ab$. By taking the square root of each side the outcome is $x = \sqrt{ab}$.

That is, $\text{G.M.} = \sqrt{ab}$.

For example: Again let $a = 4$ and $b = 9$ then the Geometric Mean would be $\text{G.M.} = \sqrt{4 \cdot 9} = \sqrt{36} = 6$.

<u>Harmonic Mean</u>

The Harmonic Mean (H.M.) of two numbers is two divided by the sum of the reciprocals of the numbers.

That is H.M. $= \dfrac{2}{\frac{1}{a}+\frac{1}{b}}$.

A simplified form of the Harmonic Mean can be found by multiplying the denominator by a common denominator of the fractions in the denominator, resulting in:

$$\frac{(2)}{\left(\frac{1}{a}+\frac{1}{b}\right)} \bullet \frac{ab}{ab} = \frac{2ab}{b+a} = \frac{2ab}{a+b}$$

The Harmonic Mean is also thought of as the reciprocal of the arithmetic mean of the reciprocals of *a* and *b*.

For example: Again let $a = 4$ and $b = 9$ then the Harmonic Mean would be

$$\text{H.M.} = \frac{2(4)(9)}{4+9} = \frac{72}{13} \approx 5.538$$

<u>The relationship of the magnitudes of the arithmetic, geometric, and harmonic means</u>

While working these simple examples that described the arithmetic, geometric, and harmonic means I noticed that when using the same values for *a* and *b*, the arithmetic mean was the largest result, the geometric mean was second largest, and the smallest was the harmonic mean. One way to understand why this occurs is to start with comparing the arithmetic to the geometric mean when $a \neq b$. Can we determine if the arithmetic means is always greater than the geometric mean?

Let *a* and *b* be any positive real numbers $a \neq b$, we will look at the case $a = b$ later in the paper.

Then $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$, by multiplying we get the result $a - 2\sqrt{ab} + b \geq 0$.

By adding $2\sqrt{ab}$ to both sides $a + b \geq 2\sqrt{ab}$.

Then divide both sides by 2, leaving $\dfrac{a+b}{2} \geq \sqrt{ab}$. So the arithmetic mean will always be greater than the geometric mean.

We can do the same thing and compare the geometric mean to the harmonic mean using the same type of proof.

Again let *a* and *b* be any positive real numbers $a \neq b$, we will look at the case $a = b$ later in the paper.

Then $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$, by multiplying, the result is $a - 2\sqrt{ab} + b \geq 0$.

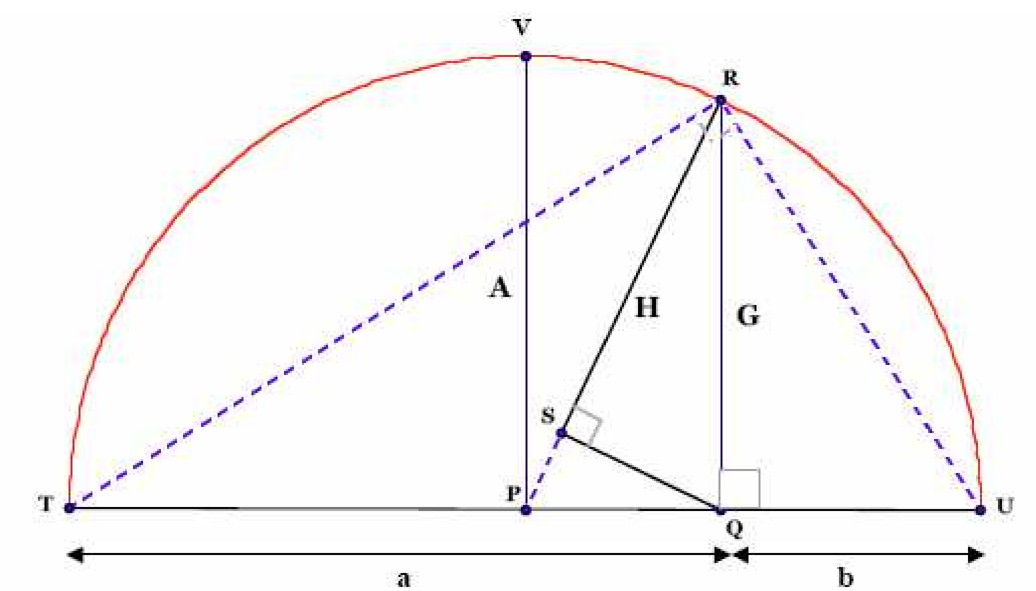
By adding $2\sqrt{ab}$ to both sides $a + b \geq 2\sqrt{ab}$.

Then divide both sides by $a + b$ leaving $1 \geq \dfrac{2\sqrt{ab}}{a+b}$.

We can then multiply both sides by $\sqrt{ab}$ to get a final result of $\sqrt{ab} \geq \dfrac{2ab}{a+b}$.

Now we know that the geometric mean is greater than the harmonic mean. By the transitive property of inequalities we know the arithmetic mean > geometric mean > the harmonic mean.

We can also see this relationship geometrically. The letters A, G, and H represent the arithmetic, geometric, and harmonic means respectively of values *a* and *b.*



The length $a+b$ represents the diameter of the semi-circle so it is easily recognized that the radius of the circle would be $\frac{a+b}{2}$. So the radius $\overline{PV}$ would represent $\frac{a+b}{2}$, the arithmetic mean. The geometric mean is an altitude of a right triangle creating two similar right triangles $\Delta TRQ \sim \Delta RUQ$. Knowing that similar triangles have corresponding sides proportional so we can create a proportion $\frac{UQ}{QR} = \frac{RQ}{QT}$.

Let x = the length of the line segment $\overline{RQ}$.
b = the length $\overline{UQ}$ and a = the length $\overline{QT}$

So $\frac{b}{x} = \frac{x}{a}$ then through the use of cross products $x^2 = ab$, which leads to $x = \sqrt{ab}$. So x = the geometric mean of a and b.

We can also use the properties of similar triangles to describe the length H. With similar triangles we know the following proportion is true $\frac{hypotenuse}{leg} = \frac{hypotenuse}{leg}$.

Let x = the length of the line segment $\overline{RQ}$
and y = the length of the line segment $\overline{RS}$
and z = the length of the radius $\overline{PR}$

Then $\frac{x}{y} = \frac{z}{x}$ and from this, since we know the geometric mean is $\sqrt{ab}$ and the radius is the same as the arithmetic mean $\frac{a+b}{2}$, we can rewrite the proportion as $\frac{\sqrt{ab}}{y} = \frac{\frac{a+b}{2}}{\sqrt{ab}}$. By using cross products $y \cdot \frac{a+b}{2} = ab$. Then multiply both sides by $\frac{2}{a+b}$ to get $y = \frac{2ab}{a+b}$. Pretty nifty the harmonic mean. So the picture really shows what the algebra proved above.

<u>When does the Arithmetic Mean equal the Geometric Mean?</u>

In order for the arithmetic mean to equal the geometric mean then $\frac{a+b}{2} = \sqrt{ab}$.
My first instinct is to clear the square root signs by squaring both sides, resulting in $\frac{(a+b)^2}{4} = ab$.

By multiplying both sides by four, $(a+b)^2 = 4ab$ and by multiplying the binomial we have $a^2 + 2ab + b^2 = 4ab$. Now subtract 4ab from both sides giving us $a^2 - 2ab + b^2 = 0$. We can rewrite this value as $(a-b)^2 = 0$.

So $a - b = 0$ and $a = b$.

Therefore if the arithmetic mean equals the geometric mean, then *a* equals *b*.

Can the converse also be true? What if we are given *a* equals *b*, can we prove that the arithmetic and geometric means are equal also?

If $a = b$ then $a - b = 0$

By squaring both sides the result is $(a-b)^2 = 0$, which leads to $a^2 - 2ab + b^2 = 0$. Now we can add $4ab$ to both sides resulting in $a^2 + 2ab + b^2 = 4ab$. Then by factoring $(a+b)^2 = 4ab$. Now we can take the square root of both sides to get $a + b = 2\sqrt{ab}$ and divide both sides by 2 for the final result of $\frac{a+b}{2} = \sqrt{ab}$. From this we can conclude the converse if $a = b$ then the arithmetic mean of *a* and *b* is equal to the geometric mean of *a* and *b*.

Although the question did not ask for the harmonic mean, as an interesting side note if $a = b$ then the same value could be entered for both values *a* and *b*. So $\frac{2}{\frac{1}{a} + \frac{1}{a}} = \frac{2}{\frac{2}{a}} = a$. If $a = b$ then the harmonic mean of *a* and *b* equals *a* and *b*.

<u>When do you use each of the Pythagorean Means?</u>

During eighth grade, students review mathematics they previously learned and than begin to make algebraic connection to the mathematics. We discuss the arithmetic mean, however in my case, the geometric and harmonic means are just faded memories if anything, and I needed to understand when it is more appropriate to use one mean over the other.

It was easy to find examples of the arithmetic mean in my own Math 8 textbook, McDougal Littell Middle School Math Course 3. Found in Lesson 5.8 Mean, Median, and Mode, these problems represent examples of some of the uses of the arithmetic mean.

1. A marine biologist records the location of deep-sea jellies in relation to the ocean surface. Jellies are found at –2278 feet, -1875 feet, -3210 feet, -2755 feet, and –2901 feet. What is the average location of a deep-sea jelly?

Solution: By using the arithmetic mean if we divide the total by the number of jellies the marine biologist locates, $\frac{(-2278)+(-1875)+(-3210)+(-2755)+(-2407)+(-2901)}{6}=\frac{-15,426}{6}=-2571.$ The mean location in relation to the ocean surface is –2571 feet.

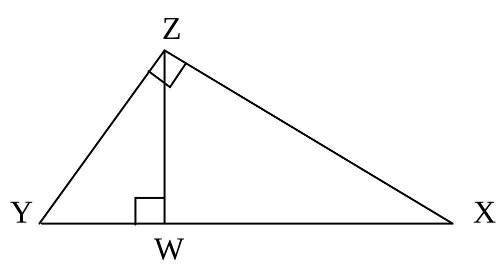
Some problems stretch my students in regular eighth grade mathematics, because of their limited algebra skills, for example:

2. You are bowling three games. In the first two games, you score 125 and 113 points. How many points do you need in the third game to have a mean score of 126 point?

Solution: Let x be the score of the third game then $\frac{125+113+x}{3}=126$. By multiplying both sides of the equation by 3 the result is $125+113+x=378$. Combining like terms simplifies the equation to $238+x=378$. Finally subtracting 238 from both sides of the equation gives us the score of the third game if we want a mean score of 126 points $x=140$.

I could not recall when students learn about the geometric mean, so since I had some Geometry and Algebra 2 textbooks at home, I started scanning the indexes and found the geometric mean as a topic in the geometry book. I decided that made sense. By using the geometric mean and their knowledge of similar triangles, the students could find the length of the altitude of right triangles.

1. Consider right triangle XYZ with altitude $\overline{WZ}$ drawn from the right angle Z to the hypotenuse $\overline{XY}$. The resulting right triangles ΔXZW and ΔZYW are similar to ΔXYZ. Since the triangles are similar, corresponding sides are proportional and $\overline{ZW}$ is the geometric mean of $\overline{XW}$ and $\overline{WY}$ (theorem).



So in ΔXYZ, if $\overline{YW}$ = 3 and $\overline{XW}$ = 14. Find $\overline{ZW}$.

Solution: If $\overline{ZW}$ is the geometric mean of $\overline{XW}$ and $\overline{YW}$ we can see this within the proportion $\frac{YW}{WZ}=\frac{WZ}{WX}$, if we let $x=WZ$ then $\frac{3}{x}=\frac{x}{14}$. Using cross products $x^2=3\cdot 14$, which means $x=\sqrt{42}$. So $WZ\approx 6.5$.

The geometric mean has mathematical uses beyond right triangles, however. The next example from the website “Ask Dr. Math” helped me understand when you would use the arithmetic mean over the geometric mean.

2. The profit from Company A, SYZO Ltd., has grown over the last three years by 10 million, 12 million, and 14 million dollars. What is the average growth during these three years?

The profit from Company B, OZYS Ltd., has grown over the last three years by 2.5%, 3%, and 3.5%. What is the average growth over during these three years?

Solution: When working with means, you are working with a bunch of different numbers. What you are trying to accomplish is to replace each of the numbers with the “same” number, without changing the result.

With the first part of the question we want to replace each of the numbers with the “same” number and get the same total. We are attempting to solve the following equation.

$$10+12+14=n+n+n$$

So by combining like term we get $36=3n$ and then $n=12$. What we are saying is that we could replace each number with 12,000,000 and get the same result as adding the three given numbers, so $12,000,000 would be the mean profit.

<u>Generalized form of the Arithmetic Mean</u>

At this point, the example above leads us to a generalized form of the arithmetic mean. If we are looking for a *n* to replace the values with the “same” value, the amount of *n*(s) could be any fixed amount, just make sure you divide by the number of *n*(s). So a generalized form of the arithmetic mean is

A.M. = $\frac{a_1+a_2+...+a_{n-1}+a_n}{n}$ where a is a real number and n is a positive integer.

In the second part of the question, if a company grows annually by 2.5%, 3%, and 3.5%, the growth rate over the three years is computed differently:

Profit in year 1 = 1.025 times the profit in year 0

Profit in year 2 = 1.03 times profit in year 1, which is 1.03 times 1.025 times the profit in year 1

Profit in year 3 = 1.035 times the profit in year 2, which is 1.035 times 1.03 times 1.025 times the profit in year 0

Thus the ratio of the profit in the third year to the profit in the base year is $1.025 \cdot 1.03 \cdot 1.035$

If the growth rate had been the same each year, the ratio would be $n \cdot n \cdot n$.

For the growth rate to be the same in both cases, we must have $1.025 \cdot 1.03 \cdot 1.035 = n \cdot n \cdot n$

So the cube root of the product of the annual growth factors is the geometric mean.

<u>Generalized form of the Geometric Mean</u>

Since it is a similar result for the geometric mean, only instead we are looking for a *n* that could replace the value with the "same" value, again the amount of *n*(s) could be any fixed amount, just may sure you take the *n*th root of the product. So a generalized form of the geometric mean is

G.M. = $\sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_{n-1} \cdot a_n}$ where *a* is a positive real number and *n* is a positive integer.

One use of the harmonic mean is finding the average of speeds over a fixed distance. For example, a car travels 300 miles at 50 mph, then travels an additional 300 miles at 75 mph. What is the average speed of the trip?

Solution: The time for the first part is $\frac{300}{50} = 6$ hours, and the time for the second part is $\frac{300}{75} = 4$ hours. So the average speed for the whole trip is $\frac{600}{10} = 60$ mph. So $\frac{2 \cdot 300}{\frac{300}{50} + \frac{300}{75}} = \frac{600}{6+4} = 60$ mph. We can also see this in the simplified version of the harmonic mean $\frac{2ab}{a+b}$ by substituting $\frac{2 \cdot 50 \cdot 75}{50 + 75} = \frac{7500}{125} = 60$ mph.

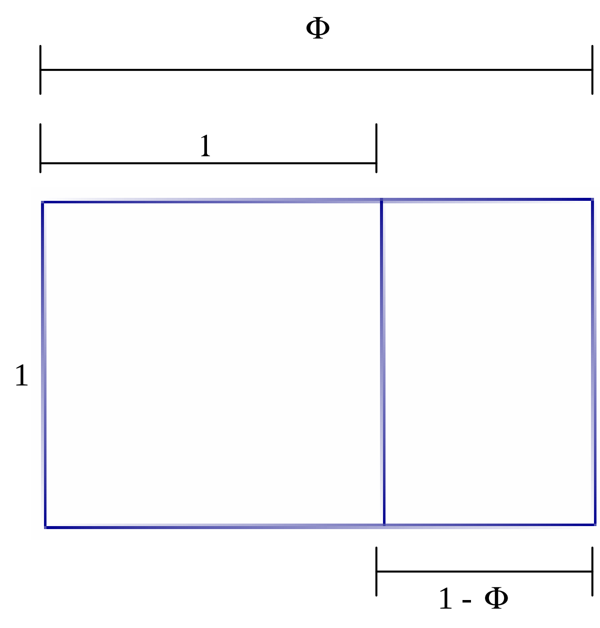
Generalized form of the Harmonic Mean

We know that the harmonic mean for two numbers *a* and *b* is the reciprocal of the arithmetic mean of the reciprocals of *a* and *b*. Since we can generalize the arithmetic mean, the harmonic mean should follow in the same manner. So a generalized form of the harmonic mean is

H.M. $(a_1, a_2, \ldots, a_{n-1}, a_n) = \dfrac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_{n-1}} + \dfrac{1}{a_n}}$ where a is a positive real number and *n* is a positive integer.

Golden Mean

The Golden Mean is also known as the golden ratio, golden section, or the divine proportion. It has fascinated mathematicians, biologists, artists musicians, and architects for its unique qualities. First written mention of the Golden Mean is in Euclid's *Element*, "If a straight line is cut in extreme and mean ratio, then as the whole is to the greater segment, the greater segment is the lesser segment." There is some evidence that the Pythagoreans knew of the Golden Ratio and perhaps that the Egyptians built the pyramids according to this proportion, but since much of this is not in written form we cannot be completely sure. In mathematical literature, the symbol used for the Golden Ratio was the Greek letter tau (τ; from the Greek τομη, which means "the cut" or "the section"). In the early twentieth century Mark Barr gave the ratio the name Φ after the Greek sculptor Phidias who was thought to have used the Golden ratio in his work.



Using the ratio that Euclid explained we could take the diagram above and set up the follow ratio.

$$\frac{\Phi}{1} = \frac{1}{\Phi - 1}$$

From this proportion, through the use of cross products we arrive at $\Phi(\Phi - 1) = 1$. Then using the distributive property, we arrive at $\Phi^2 - \Phi = 1$. By subtracting one from each side we get the quadratic equation $\Phi^2 - \Phi - 1 = 0$.

The quadratic equation can be solved by using the quadratic formula $\Phi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We know that $a = 1$, $b = -1$, and $c = -1$.

So $\Phi = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

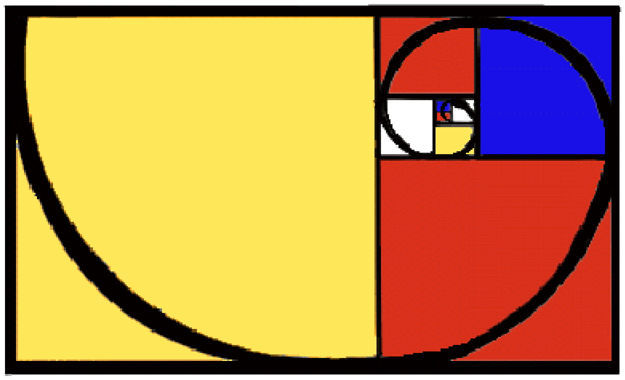
$$= \frac{1 \pm \sqrt{1 + 4}}{2}$$

$= \frac{1 \pm \sqrt{5}}{2}$ which means that $\Phi \approx 1.61803398875$ or $\Phi \approx -.61803398875$

We can now recognize that Φ is an irrational number like π or *e*, which indicates that the ratio of the two lengths cannot be expressed by a fraction, because there are no common measure other than 1 contained in the two lengths. If two lengths have no common measure they are called *incommensurable*. In Mario Livio book The Golden Ratio, on pages 4 and 5 he says,

> The Pythagorean worldview was based on an extreme admiration for the *arithmos* – the intrinsic properties of whole numbers or their ratios – and their presumed role in the cosmos. The realization that there exist numbers, like the Golden Ratio, that go on forever without displaying any repetition or pattern caused a true philosophical crisis. Legend even claims that, overwhelmed with this stupendous discovery, the Pythagoreans sacrificed a hundred oxen in awe, although this appears highly unlikely, given the fact that the Pythagoreans were strict vegetarians. . . What is clear is that the Pythagoreans basically believed that the existence of such numbers was so horrific that it must represent some sort of cosmic error, one that should be suppressed and kept secret.

The positive solution for Φ is the Golden Ratio, but if Φ is squared the result is 2.61803398875. Its reciprocal is 0.61803398875, which is the opposite of the negative solution. It is interesting to note that in order to get Φ^2 just add 1, and to get the reciprocal of Φ just subtract 1.



The Golden Mean also led to the discovery of the famous Golden Rectangle, a rectangle that through sub-division can repeat itself over and over without end. We also learned that the ratios of Fibonacci numbers converge on the Golden Ratio as we use higher and higher terms in the sequence.

Fibonacci number	Ratio	Decimal value
1		
1	$\frac{1}{1}$	1
2	$\frac{2}{1}$	2
3	$\frac{3}{2}$	1.5
5	$\frac{5}{3}$	≈ 1.67
8	$\frac{8}{5}$	1.6
13	$\frac{13}{8}$	1.625

Along with other fascinating facts, much speculation has been made also on the use of it in ancient art and architecture.

I have learned so much from the writing of this paper. It has opened my eyes to some mathematics I can do with my students. In some ways this paper is not complete, because as I come to the end, I now have more questions. For example, as I researched this paper I found that there were many other means, such as the root-mean-square, power mean, and Lehmer mean. How are these connected to the arithmetic, geometric, and harmonic mean? Beside the ones I listed, are there any other means to consider? I know that I have only seen the “tip of the iceberg” when it comes to the study of means, and this research has created a curiosity in me that will continue my drive to learn.

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