Generalizing the Pythagorean Theorem

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Introduction

Pythagoras was a mathematician and philosopher who believed mathematics was found in nature. He lived between 1600 BC and 1400 BC. While ancient Greeks had observed the relationship between the side lengths of right triangles and corresponding area, the concept never appeared in print until Euclid. Pythagoras opened a school of religion and mathematics. The work of Pythagoras himself or work produced by others at the school has not been distinguished. Either Pythagoras or a student’s of his actually proved the famous Pythagorean Theorem, which asserts that given a right triangle, the sum of the squared side lengths equals the square of the hypotenuse. Most of his work was geometrical as pertained to nature. Please note that throughout this paper I am using the typical variables of a triangle with vertices A, B and C and side lengths a, b, and c accordingly where c is the hypotenuse.

Proofs of the Pythagorean Theorem

I will show three out of hundreds of existing proofs for the Pythagorean Theorem. The first two were explored through courses taken while participating in Math in the Middle.

Figure 1 shows four right triangles in a rectangle with a total area of 2ab. These four triangles were rearranged on the right to form a square within a square as seen in Figure 2. The inner square has a side length equal to the hypotenuse of the right triangles, c and an area of $c^2$. 
The outer edge of the square consists of both leg $a$ and leg $b$ of the triangle. Each side measures $a+b$ yielding an area of $(a + b)^2$ on the outside with length $c$. The actual area of the triangles, which is known from the rectangle to equal $2ab$ from Figure 1, can be found by subtracting the area of the smaller inner square from the larger outside square and obtain the Pythagorean Theorem.

\[
(a + b)^2 - c^2 = 2ab \\
(a + b)^2 + b^2 - c^2 = 2ab \\
a^2 + 2ab + b^2 = 2ab + c^2 \\
a^2 + b^2 = c^2
\]

Once again, we start with the four right triangles in Figure 1 with a total area of $2ab$, and we can rearrange these to form a square with a square space in the center. Figure 3 shows the result of inverting each triangle from Figure 2. The hypotenuse is now the length of the outer square with area of $c^2$. The inner square has a side length that is the positive difference in lengths of the legs of the triangle. Note that if the triangle is an isosceles right triangle the inner square would not be seen and it would have an area of 0. Let $b > a$ so the length of the inner square is $(b-a)$ and the area is $(b-a)^2$. Again taking the area of the larger square and subtracting the area of the inner square, which equals $2ab$, simplifies to yield the Pythagorean Theorem.

\[
c^2 - (b-a)^2 = 2ab \\
c^2 - (b^2 - 2ab + a^2) = 2ab \\
c^2 - b^2 + 2ab - a^2 = 2ab \\
c^2 + 2ab = a^2 + b^2 + 2ab \\
c^2 = a^2 + b^2 \\
a^2 + b^2 = c^2
\]
The third proof I found interesting starts with a right triangle ABC in Figure 4 below and constructs the altitude from the right angle to point D on the hypotenuse. The sides of the original triangle are \( a = BC \), \( b = AC \), and \( c = AB \). Now three similar triangles are present. The ratios of corresponding sides of the two smaller right triangles to the original are equated and simplify to the Pythagorean Theorem.

Now, triangle \( BDC \sim \triangle CDA \sim \triangle BCA \). Since these are similar triangles the sides are proportional. Comparing the triangle on the right to the original \( \frac{BD}{BC} = \frac{BC}{AB} \). Looking at the triangle on the right to the original \( \frac{AD}{AC} = \frac{AC}{AB} \). Using cross products on each equation results with the equations \( (BC)(BC) = (AB)(BD) \) and \( (AC)(AC) = (AD)(AB) \). Using the addition property of equality helps combine the equations into one statement. Simplifying and using segment addition yields the Pythagorean Theorem once again.

\[
(BC)(BC) + (AC)(AC) = (AB)(BD) + (AD)(AB)
\]

\[
(BC)^2 + (AC)^2 = (AB)(BD + AD)
\]

\[
(BC)^2 + (AC)^2 = (AB)(AB)
\]

\[
(BC)^2 + (AC)^2 = (AB)^2
\]

\[
a^2 + b^2 = c^2
\]

**Law of Cosines**

The Law of Cosines is a theorem used to help find angle measures and missing side lengths of triangles. The easiest calculation with the Law of Cosines is to look for a missing side length when two side lengths and the angle measure between the sides is known. Unlike the
Pythagorean Theorem, the Law of Cosines is applicable to more than right triangles; it may also be applied both to acute and to obtuse triangles, as shown in figure 5 below. When a right triangle is considered, the Law of Cosines formula simplifies to Pythagorean Theorem. Without loss of generality, let angle C be the right angle so it has a measure of 90 degrees. From trigonometry we know the \( \cos 90^\circ \) (degrees) is zero.

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

\[
c^2 = a^2 + b^2 - 2ab \cos 90^\circ
\]

\[
c^2 = a^2 + b^2 - 2ab(0)
\]

\[
c^2 = a^2 + b^2 - 0
\]

\[
c^2 = a^2 + b^2
\]

Therefore the Law of Cosines is a generalization of the Pythagorean Theorem.

**Generalizing by building on a right triangle**

The general picture that serves as a mnemonic device for many is a right triangle with squares on each side as in Figure 6 below.
Replacing the squares with equilateral triangles on each side, as shown in Figure 7 below, the Pythagorean Theorem will still hold true.

The area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$. In order to find the height of the triangle, construct or draw the perpendicular bisector to the side of the right triangle. Now we see two $30 - 60 - 90$ triangles whose height is equal to $(.5)(\sqrt{3})(\text{hypotenuse})$. Now the total area for each triangle can be found:

$$\frac{1}{2} (\text{side length})((.5)(\sqrt{3})(\text{side length})).$$

$$\frac{1}{4}(\sqrt{3})(\text{side length})^2$$

We can then substitute $a$, $b$, and $c$, for the side lengths to prove that the sum of the two smaller areas equal to the largest area is the Pythagorean Theorem.

$$\frac{1}{4}(\sqrt{3})(a)^2 + \frac{1}{4}(\sqrt{3})(b)^2 = \frac{1}{4}(\sqrt{3})(c)^2$$

$$\frac{1}{4}(\sqrt{3})(a^2 + b^2) = \frac{1}{4}(\sqrt{3})(c^2)$$

$$a^2 + b^2 = c^2$$
Figure 8 above shows a regular hexagon one each side of the right triangle. The next shape I tried to test my hypothesis was to have a hexagon. This time I just started with the area of regular polygons recalling the formula that the area equals one-half times the apothem times the perimeter. In this case the apothem is the perpendicular distance from the center of a polygon to the edge. Also the apothem is the same as the height of the equilateral triangles, drawn from the center to each vertex of the polygon. The perimeter would equal the number of sides times each side length. Each hexagon has an area equal to
\[ \frac{1}{2}(\frac{\sqrt{3}}{2})(\text{side length})(6(\text{side length})). \]
\[ \frac{3}{2}(\sqrt{3})(\text{side length})^2 \]

Again I see the numerical coefficient of the squared side length which will be the same for each side and this will show the Pythagorean Theorem also. I felt confident that realistically the Pythagorean theorem could be stated for all regular polygons.

I recalled that during last school year I showed something similar to an Algebra 1 class. I had assigned a critical thinking problem from the textbook that asked them to show the Pythagorean Theorem was also true if semi-circles were on each side -- see figure 9 below and note the values are rounded. Semi-circles are not regular polygons but are in some sense the limit of a sequence of polygons as the number of sides tends to positive infinity. The area for a
semi-circle is $\frac{1}{2}(\pi)(\text{radius}^2)$. The radius for each semi-circle is $\frac{1}{2}(\text{side length})$ and when that is squared the area is expressed as

$$\frac{1}{2}(\pi)(\frac{1}{2} \text{(side length)})^2$$

$$= \frac{1}{8}(\pi)(\text{side length}^2)$$

So if the two smaller areas sum to equal the larger, then the Pythagorean Theorem prevails.

$$\frac{1}{8}(\pi)a^2 + \frac{1}{8}(\pi)b^2 = \frac{1}{8}(\pi)c^2$$

$$\frac{1}{8}(\pi)(a^2 + b^2) = \frac{1}{8}(\pi)c^2$$

$$a^2 + b^2 = c^2$$

Consider Isosceles Triangles

The next case was a challenge and a revelation. I knew that the Pythagorean Theorem could work with any regular shape. For the polygons I could always find the angle and use $\frac{1}{4}(\tan x)(\text{side length})^2$. This was my strategy when I considered octagons. However if a shape was not a regular polygon, then the angle could not be found. Also without the angle, I would not be able to find the height. The smallest irregular polygon is an isosceles triangle. I was struggling to decide which side length would be different. Eventually I decided to have the side of the right triangle be the base of the isosceles triangle. For quite a while I kept on stumbling on a barricade of too many variables. On the first day of our Capstone Course, trigonometry was introduced as related to the study of the side lengths and associated angle measures of right triangles; this, of course, is related to the concept of similarity in geometry. This is when I realized the isosceles triangles would work if I defined them to be similar then their angles would
be the same and the height would be \( \frac{1}{2} \) (side length)(tan x). Therefore the heights would be proportional to the base to obtain the squared side lengths.

\[
\frac{1}{2}(a)(\frac{1}{2}a \cdot \tan x) + \frac{1}{2}(b)(\frac{1}{2}b \cdot \tan x) = \frac{1}{2}(c)(\frac{1}{2}c \cdot \tan x)
\]
\[
\frac{1}{2} \tan x(a^2) + \frac{1}{2} \tan x(b^2) = \frac{1}{2} \tan x(c^2)
\]
\[
\frac{1}{2} \tan x(a^2 + b^2) = \frac{1}{2} \tan x(c^2)
\]
\[
a^2 + b^2 = c^2
\]

This could also be generalized through the regular polygons. Drawing the segment from the center of a regular polygon to the vertices usually will produce multiple isosceles triangles.

The Pythagorean Theorem can be generalized for any similar triangles built off the sides of a right triangle. Figure 11 below shows three similar triangles.
The big picture is that given sides of a right triangle, the sum of similar areas built on the two shorter sides will equal the area built on the longest side. All similar shapes are proportional or have a comparative scale factor. All corresponding angles are congruent as shown in Figure 11. The corresponding side lengths are proportional with the scale factor while their areas are proportional to the squared scale factor since area in two-dimensional. Compare the smaller shapes built off sides a and b of a right triangle to the largest side c by dividing each side length by c to find the scale factor. Then square each ratio to find a relative area per side. If the two smaller areas sum the largest, then once again the Pythagorean Theorem emerges as proof that the sum of the side lengths squared also sum the square of the largest side length for all similar triangles and other similar shapes.

\[
\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \left(\frac{c}{c}\right)^2
\]

\[
\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1
\]

\[
a^2 + b^2 = c^2
\]

Knowing the Fundamental Theorem of Calculus could be used to find areas of more complex shapes given a function, I drew half a cycle of the sine curve over a distance of 3, 4 and 5 units from the origin. Then I integrated the functions to find the area above the x-axis. Figure 12 shows that the functions were altered to allow for a change in period and amplitude.

![Figure 12](image)

The Pythagorean Theorem would work in situations resembling Figure 13, given that the ellipses have the same eccentricity value and other situations where the shapes built on each side of the right triangle are similar.
Conclusions

Ultimately the Pythagorean theorem states that if the squared distances between corresponding points of two similar figures sum to equal another squared distance, so will their corresponding areas. The Pythagorean Theorem can be applied to show that the points corresponding area and linear distance are proportional. The Pythagorean Theorem is very foundational and has many applications well beyond the picture of squares drawn on a right triangle to assist students to recall the formula.
References

http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Pythagoras.html