Function Composition and Linear Fractional Transformations

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Function composition and linear fractional transformations may not appear to be connected to each other at first glance, but they do contain many relationships. These mathematical topics and their relationships with each other are investigated in this paper. Within our investigation, we will discuss the definition of function composition, consider requirements which must be met for function composition, define linear fractional transformations, and analyze behaviors of linear fractional transformations. We then will delve into composition of linear fractional transformations and demonstrate how to generate lists of functions using ideas from compositions. Finally, we will connect the linear fractional transformation composition to matrix multiplication and see a surprising result.

**Function Composition**

We first recall the definition of a function. While there are a number of competing definitions of a function, for our purposes we will use the following: a function is the mathematical concept which expresses the relationship between two quantities, one of which is given as an independent variable or sometimes called the input and the other of which is produced as the dependent variable or sometimes referred to as the output. The function acts upon the input with some operation to change it into the output. Because this operation does not change over time, each input always corresponds to exactly one output. According to Mathworld.com, function composition “is the nesting of two or more functions to form a single new function.” To put this in more
elementary terms, the composition of functions is the function obtained by taking the output from one function and using that as the input for a second function.

There are two different ways to denote function composition. The first is to use the notation of \( f \circ g \). Although this looks like the word fog, it is read “f of g(x)” or “f of g of x”. What this really means is for each value of the input \( x \), the function \( g \) yields an output called \( g(x) \). Then, this output \( g(x) \) is put into our second function \( f \). This then yields the output for the function \( f \circ g \). This can also be written as \( f(g(x)) \) which has the same meaning. The latter notation is often preferable because it shows that the output of \( g(x) \) is the input for \( f(x) \). An example of function composition would be: If \( f(x) = 3 \sin(5x) \) and \( g(x) = x^2 \), then \( f(g(x)) = 3 \sin(5x^2) \) and \( g(f(x)) = [3 \sin(5x)]^2 = 9 \sin^2(5x) \).

There are some conditions that must be met for function composition to work. For \( f(g(x)) \) to be defined it is necessary that the set of outputs (or image) of \( g(x) \) be contained in the domain of \( f(x) \). The domain of the composition of \( f(g(x)) \) must be a subset of the domain of \( g(x) \) and the domain of the composition of \( f(g(x)) \) must only include the values of \( x \) which \( g(x) \) maps into the domain of \( f(x) \).

We cannot merely look at the formula for the composition of \( f(g(x)) \) to find the domain and range of the composition but we need to consider the whole process of the composition. For instance, consider the case where \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \). The composition of \( f(g(x)) \) is \( x \), which would appear to have the domain and range of all real numbers but the domain of \( g(x) \) is only positive real numbers which when put into the function of \( f(x) \) would yield a range of only positive real numbers. So, in this case, we can see we need to consider carefully what the domain and range of a composition will be.
Another concept to consider when discussing function composition is inverse functions. The inverse of a function is closely related to the idea of the identity function. The identity function is the function which gives the same output as the input. This function is \( f(x) = x \). If we apply this idea to composition of functions, we can find inverses for \( f(x) \). For instance, if we find the inverse of \( f(x) \) (assuming it exists) and call it \( g(x) \) and find the compositions \( f(g(x)) \) and \( g(f(x)) \), we arrive at the identity function; that is \( f(g(x)) = g(f(x)) = x \).

Note that many functions do not have an inverse function. For example, consider \( f(x) = x^2 \). There are two numbers that we can input into \( f(x) \) and get an output of 4, \( f(2) = 4 \) and \( f(-2) = 4 \). If \( f(x) \) had an inverse, then the fact that \( f(2) = 4 \) would imply that the inverse of \( f(x) \) takes 4 back to 2. On the other hand, since \( f(-2) = 4 \), the inverse of \( f(x) \) would have to take 4 to -2. This is a problem because 2 is not equal to -2. Therefore, there is no function that is the inverse of \( f(x) \).

If we look at the same problem in terms of graphs, we can get a better idea of what we are talking about. If \( f(x) \) had an inverse, then its graph would be the reflection of the graph of \( f(x) \) about the line \( y = x \). The graph of \( f(x) \) and its reflection about \( y = x \) are drawn below.
Note that the reflected graph does not pass the vertical line test (i.e. there is more than one output corresponding to a given input), so it is not the graph of a function. Therefore, \( f(x) \) does not have an inverse that is a function.

To generalize this, we can say that a function \( f(x) \) has an inverse if and only if when its graph is reflected about the line \( y = x \), the result is the graph of a function (passing the vertical line test). But this can be further simplified. We can tell before we reflect the graph whether or not any vertical line will intersect the reflected graph more than once by looking at how horizontal lines intersect the original graph! So, if the original function \( f(x) \) does not pass a “horizontal line test”, i.e. there exists a horizontal line that intersects the graph of \( f(x) \) at more than one point, then there is no inverse for this function. Another way of saying this is that a function which passes the horizontal line test is one-to-one (injective), and therefore has an inverse.

Another important fact about the composition of functions has to do with the associative property. Composition of functions is always associative. That is, if \( f(x) \), \( g(x) \), and \( h(x) \) are three functions with suitably chosen domains and ranges, then \( f \circ (g \circ h) = \).
(f \circ g) \circ h = f \circ (g \circ h) = f(g(h(x))). Even though function composition is associative, function composition is not generally commutative, as we will demonstrate.

**Linear Fractional Transformations**

Linear fractional transformations are functions of the form \( f(x) = \frac{ax + b}{cx + d} \), where \( a, b, c, \) and \( d \) are real numbers and satisfy the condition that \( ad - bc \neq 0 \). There are four types of behaviors that we see for different values of \( a, b, c \) and \( d \). These types of behaviors were found by checking different values of \( a, b, c \) and \( d \) and then analyzing the behavior of the graphs on the interval \([-1, 1]\) and sometimes the interval \([1, 2]\). All examples were examined to see if they fit the form \( f(x) = \frac{ax + b}{cx + d} \), where \( a, b, c \) and \( d \) are real numbers and satisfy the condition that \( ad - bc \neq 0 \).

The first type of behavior is scaling. The simplest example of a scaling linear fractional transformation would be \( f(x) = \frac{1x + 0}{0x + 1} = x \). This is the identity function, which we have seen before. Another scaling example where we can see more of the scaling effect is \( f(x) = \frac{3x + 0}{0x + 1} = 3x \). This transforms the interval \([-1, 1]\) to \([-3, 3]\). Here we see the transformation scaling the interval by a factor of 3. The basic form of the scaling linear fractional transformation is \( f(x) = ax \) where \( a \in \mathbb{R} \) with \( a \neq 0 \), \( b = 0 \), \( c = 0 \) and \( d = 1 \).

Another form of scaling behavior that adds a twist is reflection. Reflection happens when the value of \( a \) is negative in our general form of the scaling linear
fractional transformation. For example, the simplest case would be

\[ f(x) = -\frac{1x + 0}{0x + 1} = -x \quad \text{or} \quad f(x) = \frac{1x + 0}{0x - 1} = -x. \]

If we consider the effect of this transformation on the interval \([1, 2]\), we see this interval transformed to \([-2, -1]\). This transformation “flips” the original interval about the value \(x = 0\). This is what we call a reflection. We can also have a reflection with a larger scaling such as the example \(f(x) = \frac{3x + 0}{0x - 1} = -3x\). This will transform the interval \([1, 2]\) to \([-6, -3]\) where 1 is transformed into -3 and 2 is transformed into -6. So, we see the reflection about the value of \(x = 0\) and the scaling is by a factor of 3. The basic form of a reflection is \(f(x) = ax\) where \(a < 0\).

The second behavior identified is translation. An example of a translation is \(f(x) = \frac{1x + 1}{0x + 1} = x + 1\). If we consider the interval \([-1, 1]\), we see it transformed to \([0, 2]\) by this transformation. This would correspond to a movement to the right on the number line by adding one. If we consider \(f(x) = \frac{1x - 1}{0x + 1} = x - 1\), this would move the interval \([-1, 1]\) to \([-2, 0]\); this is a movement to the left on the number line by subtracting 1. This movement is called a translation. The general form of a translation is \(f(x) = x + b\), where \(a = 1\), \(b \in \mathbb{R}\), \(c = 0\) and \(d = 1\).

The third behavior is inversion. An example of an inversion is

\[ f(x) = \frac{0x + 1}{1x + 0} = \frac{1}{x} \]

which transforms the interval \([-1, 1]\) into two intervals \((-\infty, -1]\) and \([1, \infty)\) with \(f(0)\) undefined (or defined by \(f(0) = \infty\)). If we consider the interval \([1, 2]\), it is
transformed to \([1/2, 1]\). So, we see 1 transformed into a 1 and 2 transformed into a \(\frac{1}{2}\). So, we see the interval transformed by flipping and shrinking.

Another interval to consider is \([1/2, 1]\) which is transformed back to \([1, 2]\). We still see the transformation flipping the interval, but now it is stretching it out. We notice that any intervals contained within \((0, 1)\) when inverted will be stretched out to an interval which is greater than one and any intervals between \((1, \infty)\) will be shrunk down to an interval between zero and one. The general form of an inversion transformation is \(f(x) = \frac{1}{x}\) where \(a = 0\), \(b = 1\), \(c = 1\) and \(d = 0\).

The fourth behavior is a combination of any of the previous three. We can combine any of the previous behaviors at once to generate a wide variety of transformations. We can do this in terms of function composition. If we combine a scaling behavior with a translation behavior we see the effects of both behaviors. For example, \(f(x) = \frac{3x + 1}{0x + 1} = 3x + 1\) transforms the interval \([-1, 1]\) to \([-2, 4]\). We first see the scaling effect with multiplying by 3 and then the translation effect of adding one by composing these two transformations. The general form of this would be \(f(x) = ax + b\), where \(a \in \mathbb{R}\) with \(a \neq 0\), \(b \in \mathbb{R}\), \(c = 0\) and \(d = 1\).

We now see an example of the fact that function composition need not be commutative. Composing these transformations in the other order results in a different transformation. In other words, if we do the translation first and then the scaling, our interval will be changed in a different way; for example,
consider \( f(x) = \frac{3(x + 1)}{0x + 1} = 3(x + 1) \). This transforms our interval from \([-1, 1]\) to \([0, 6]\). We now see the translation happening first and then the scaling.

We can also combine scaling and inversion. For example,

\[
f(x) = \frac{0x + 3}{1x} = \frac{3}{x}
\]

transforms the interval \([-1, 1]\) into two new intervals of \((-\infty, -3]\) and \([3, \infty)\) with a break in the interval at \(x = 0\). In this transformation, we can see both the effects of the inversion and the scaling. Notice that \(f(x)\) can be written as the composition \(g(h(x))\) where \(g(x) = 3x\) and \(h(x) = \frac{1}{x}\). The general form of a scaling and inversion is

\[
f(x) = \frac{b}{cx + d}\]

where \(a = 0, b \in \mathbb{R}\) with \(b \neq 0, c \in \mathbb{R}\) with \(c \neq 0\) and \(d \in \mathbb{R}\). We can also combine inversion with translation, for example \(f(x) = \frac{1x + 0}{1x + 1} = \frac{x}{x + 1}\). This transforms the interval \([-1, 1]\) into a new interval \((-\infty, \frac{1}{2}]\) and with \(f(-1)\) undefined. So in this case, we still see the inversion, but we now see a translation of the pole of the transformation to \(x = -1\).

A similar approach could be used to investigate more complicated combinations of linear fractional transformations. Note that changing the value of \(d\) in a purely scaling transformation will change the scaling factor from \(a\) to a scaling factor of \(a/d\). Note that changing \(d\) from \(d = 1\) in a pure translation will result in a combination of scaling and translation by a factor of \(1/d\). For an inversion, the pole is given by \(-d/c\), so changing the value of \(d\) will dramatically affect any transformation involving an inversion. So, we can see that we need to consider carefully what values are selected to obtain a desired linear
fractional transformation and that these values can have different effects depending upon the combination or composition of behaviors that we have.

**Composition of Linear Fractional Transformations**

Now, we will consider the possible compositions of two (carefully chosen) linear fractional transformations. Let \( f(x) = -x \) and \( g(x) = \frac{x+1}{-x+1} \). In the function \( f(x) \), we can see this is a linear fractional transformation by expanding it to \( f(x) = \frac{-1x + 0}{0x + 1} \) and we also see \( ad - bc \neq 0 \) because \( ad-bc = -1 \). In the function \( g(x) = \frac{x+1}{-x+1} \), we can see this is a linear fractional transformation by expanding it to \( g(x) = \frac{1x + 1}{-1x + 1} \) and we also see \( ad - bc \neq 0 \) because \( ad-bc = 2 \).

There are eight different functions that can be obtained by repeated compositions of \( f(x) \) and \( g(x) \) with themselves or each other. One can demonstrate that each composition below is a linear fractional transformation by checking that it is of the form

\[
f(x) = \frac{ax + b}{cx + d},
\]

where \( a, b, c, \) and \( d \) are real numbers and satisfy the condition that \( ad - bc \neq 0 \).

1. \( f(f(x)) = x \)
2. \( f(f(f(x))) = -x \), this is the original function \( f(x) \)
3. \( g(g(x)) = -\frac{1}{x} \)
4. \( g(g(g(x))) = \frac{x-1}{x+1} \)
5. \[ g(g(g(g(x)))) = \frac{x+1}{-x+1}, \text{ this is the original function } g(x) \]

6. \[ f(g(x)) = \frac{-x-1}{-x+1} \]

7. \[ g(f(x)) = \frac{-x+1}{x+1} \]

8. \[ f(g(g(x))) = \frac{1}{x} \]

To show that the above list is exhaustive, we need to use the idea of an identity. We know \( f(f(x)) = x \) is the identity function. We know that once a composition yields the identity, the composition can be simplified. Also, \( g(g(g(g(x)))) = x \) is the identity function. So, from this we know that in our composition search, we never need more than two \( f(x) \)'s composed together or more than four \( g(x) \)'s composed together.

Now, let's look at combining one \( f(x) \) with different numbers of \( g(x) \)'s. Here is a list of the start of all the possible cases: \( f(g(x)), g(f(x)), f(g(g(x))), g(g(f(x))), f(g(g(f(x)))) \) and \( g(g(g(f(x)))) \). Note that \( f(g(x)) = \left( \frac{x+1}{-x+1} \right) = \frac{-x-1}{-x+1} \) and

\[
g(g(g(f(x)))) = g(g(-x)) = g\left( g\left( \frac{-x+1}{x+1} \right) \right) = g\left( \frac{-x+1}{x+1} + 1 \right) = g\left( \frac{-x+1}{x+1} + \frac{x+1}{x+1} \right) = g\left( \frac{-x+1}{x+1} + \frac{x+1}{x+1} \right) = g\left( \frac{-x+1}{x+1} + \frac{x+1}{x+1} \right) = g\left( \frac{-x+1}{x+1} + \frac{x+1}{x+1} \right) \]

\[
g\left( \frac{2}{x+1} \right) = g\left( \frac{2 \cdot x + 1}{2x} \right) = g\left( \frac{1 + \frac{1}{x}}{2x} \right) = g\left( \frac{1 + \frac{1}{x}}{2x} \right) = g\left( \frac{1 + \frac{1}{x}}{2x} \right) = g\left( \frac{1 + \frac{1}{x}}{2x} \right) = g\left( \frac{1 + \frac{1}{x}}{2x} \right) = g\left( \frac{1 + \frac{1}{x}}{2x} \right) \]

\[
= \frac{-1-x}{1-x} = \frac{-x-1}{-x+1} \]. Thus, we know that \( f(g(x)) = \frac{-x-1}{-x+1} = g(g(g(f(x)))) \), so we can eliminate \( g(g(g(f(x)))) \) from our list of new compositions.
So, anytime we have \( f(g(x)) \) or \( g(g(f(x))) \), we can substitute one for the other. In our starting list, we can find equivalent expressions for the rest of our list in a similar manner as above. Other equivalent expressions are \( g(f(x)) = f(g(g(x))) \) and \( f(g(g(x))) = g(ff(x)) \). So, we can now also eliminate \( f(g(g(x))) \) and \( g(g(f(x))) \) from our list.

Now we need to be able to rule out the cases where we could have \( g(x) \)'s composed on both sides of \( f(x) \) and also consider the cases where \( f(x) \) could be composed on both sides of \( g(x) \)'s. For example, suppose that we have \( f(g(f(x))) \). We know that \( g(f(x)) = f(g(g(x))) \) from before, so if we substitute \( f(g(g(x))) \) in for \( g(f(x)) \) in our composition, we have \( f(f(g(g(x)))) = g(g(x)) \) because \( f(f(x)) \) is the identity function and yields \( x \). We must remember that function composition is associative, so we are allowed to associate any two compositions which are next to each other. So, \( f(g(f(x))) = f(f(g(g(x)))) = g(g(x)) \) and can now be taken off our list of possibilities.

In fact, composing \( f(x) \) with any of the previously listed compositions in either order would result in creating an identity function in part of the composition and thus reduce the composition to a previous function. So, no new compositions result by composing \( f(x) \) in front of or after one of the compositions we have already listed.

The only other cases we need to consider would be to compose \( g(x) \) with any of the previously listed compositions either before or after. If we look at composing \( g(x) \) after \( g(f(x)) \) we would have \( g(f(g(x))) \). To check to see if this is a new composition we know that \( g(f(x)) = f(g(g(x))) \), so we can substitute \( f(g(g(x))) \) into the composition for \( g(f(x)) \) to have \( f(g(g(g(x)))) \). We know that \( g(g(g(x))) = x \) or is the identity function,
so we have \( f(g(g(g(g(x))))) = f(x) \). This is already a function listed, so we can see that \( g(f(f(g(x)))) \) did not yield a new function.

In fact, composing \( g(x) \) with anything on our list would result in creating an identity function in part of the composition and thus reduce the composition to a previously listed function. So, no new compositions result by composing \( g(x) \) in front of or after one of the compositions we have already listed.

In our search for all of the functions from the composition of \( f(x) \) and \( g(x) \), we can see the utility of the concept of an identity. Because of the identity function we can rule out any of the possibilities where we would be adding a composition of an \( f(x) \) or \( g(x) \) onto any of the previous compositions that we had. This is because in all cases an identity of \( f(f(x)) \) or \( g(g(g(g(x)))) \) can be found with substitution of an equivalent composition. Then after substitution, our new composition can then be simplified to an existing function on our previous list of eight. From all this information, we know that we have found them all.

**Matrix Representations of Linear Fractional Transformations**

Linear fractional transformations have an interesting connection with real \( 2 \times 2 \) matrices. If we expand our original functions of \( f(x) \) and \( g(x) \) to see the linear fractional transformations, we have \( f(x) = \frac{-1x + 0}{0x + 1} \) and \( g(x) = \frac{1x + 1}{-1x + 1} \). If we place the coefficients of each function into a \( 2 \times 2 \) matrix, we have a matrix for \( f(x) \) of \( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \) and a matrix for \( g(x) \) of \( \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \). If we multiply these two matrices together, we have
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & -1 \\
-1 & 1
\end{bmatrix}.
\]
Placing the coefficients back into a linear fractional transformational function, we have \( \frac{-x-1}{-x+1} \) which is \( f(g(x)) \). There is a direct correspondence between composition of linear fractional transformations and multiplication of their matrix representatives.

To demonstrate this, we need to remember that any time we compose two linear fractional transformations that the composition is also a linear fractional transformation. This corresponds to matrix multiplication as well. Any time a \( 2 \times 2 \) matrix with nonzero determinant is multiplied by another \( 2 \times 2 \) matrix with nonzero determinant, the product matrix will be a \( 2 \times 2 \) matrix with nonzero determinant (this corresponds to \( ad - bc \neq 0 \)).

We know that we can find the value of the product matrix’s determinant by multiplying the determinants of the original two factor matrices. Since each of the factor matrices determinants were not equal to zero, then the product of them cannot be zero as well.

In our previous discussion, we generated a list of all of the function compositions of \( f(x) = \frac{-x+1}{x} \) and \( g(x) = \frac{x+1}{-x+1} \), we can demonstrate this in another way using matrix multiplication. The matrix multiplication of \( f(f(x)) \) is \( \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \), which corresponds to the linear fractional transformation of \( f(f(x)) = \frac{1x+0}{0x+1} = x \). The matrix multiplication of \( f(f(f(x))) \) is the matrix multiplication of \( f \circ (f \circ f)(x) \) which is \( \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} \) which corresponds to the linear fractional transformation
\[ f(f(f(x))) = \frac{-1x + 0}{0x + 1} = -x. \] This is the original function \( f(x) \), which makes sense because anytime we have two \( f(x) \)'s composed together we have the identity function.

The next composition we consider is \( g(g(x)) \). Its corresponding matrix multiplication would be
\[
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 2 \\
-2 & 0
\end{bmatrix}
\]
which corresponds to the linear fractional transformation of \( g(g(x)) = \frac{0x + 2}{-2x + 0} = -\frac{2}{2x} = -\frac{1}{x} \). We can note that the corresponding matrix for \(-\frac{1}{x}\) is \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.\]
From this we deduce that any matrix which has the form of \[
\begin{bmatrix}
0 & k \\
-k & 0
\end{bmatrix}
\]
can be simplified to a linear fractional transformation with matrix \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \]
i.e.,

to the linear fractional transformation \(-\frac{1}{x}\).

Another matrix which would give the same linear fractional transformation is
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]
and thus any matrix of the form \[
\begin{bmatrix}
0 & -k \\
k & 0
\end{bmatrix}
\]
would also simplify to the linear fractional transformation \(-\frac{1}{x}\). It is important to note that other matrices will also be able to be simplified, such as the matrix \[
\begin{bmatrix}
k & 0 \\
0 & k
\end{bmatrix}
\]
which would simplify to the identity matrix \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
which corresponds to the identity function. Also the matrices \[
\begin{bmatrix}
k & 0 \\
0 & -k
\end{bmatrix}
or
\begin{bmatrix}
-k & 0 \\
0 & k
\end{bmatrix}
\]
would simplify to \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
or \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix},
\]
which correspond to the same function.
$f(x) = -x$. Other matrices of the form $\begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$ or $\begin{bmatrix} k & -k \\ k & k \end{bmatrix}$ or $\begin{bmatrix} -k & -k \\ k & -k \end{bmatrix}$ would simplify to linear fractional transformations with matrices $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ respectively, which correspond to the linear fractional transformations $\frac{-x+1}{x+1}$, $\frac{x-1}{x+1}$, $\frac{x+1}{-x+1}$ and $\frac{x+1}{x-1}$, respectively. We will show how we would arrive at these linear fractional transformations next with matrix multiplication.

The next function we consider is $g(g(x)) = g \circ g \circ g(x)$, which corresponds to the matrix multiplication $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. This corresponds to the linear fractional transformation of $\frac{-1x+1}{-1x-1} = \frac{-x+1}{-x-1}$. If we multiply this by -1 in the numerator and denominator, we arrive at $\frac{x-1}{x+1}$ which is the function $g(g(g(x)))$. If we consider $g(g(g(g(x)))))$ in a similar manner, we see $g(g(g(g(x)))) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is the identity matrix, thus $g(g(g(g(g(x))))) = g(x)$.

Next we examine $f(g(x))$. This composition has the matrix representation

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ which corresponds to the linear fractional transformation $f(g(x)) = \frac{-x-1}{-x+1}$. 
Another function for which we can show the corresponding matrix multiplication is

\[ g(f(x)) : \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \]

which corresponds to the linear fractional transformation of \( g(f(x)) = \frac{-x + 1}{x + 1} \). The last function in our list of eight is \( f(g(g(x))) \)

which would be the matrix multiplication for \( f \circ (g \circ g)(x) : \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

This matrix \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) corresponds to the linear fractional transformation \( f(g(g(x))) = \frac{1}{x} \).

From all of the previous examples of matrix multiplication, we have found all eight different functions which can be obtained by composing \( f(x) \) and \( g(x) \) using matrix multiplication. We know that this is all of them because of the simplifications that we were able to make in the previous discussion.

**Conclusion**

In our search to learn about linear fractional transformations, we have defined function composition and behaviors for linear fractional transformations; connecting combinations of behaviors with composition. Then we explored function composition by looking at the repeated composition of two specific linear fractional compositions. In our exploration, we found all possible function compositions and looked at the concepts of identity and equivalent expressions to help us show by exhaustion that our list was complete. The last part of this paper connected function composition to matrix multiplication. This connection, although not obvious at first, allows another way to demonstrate function composition of linear fractional transformations. From all of this
investigation, we can better understand functions in general and gain an appreciation for function composition and linear fractional transformations.
References


http://mathworld.wolfram.com/Composition.html

http://oregonstate.edu/instruct/mth251/cq/FieldGuide/composition/lesson.html