The Mathematics of the Five Card Trick

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Introduction

Mathematics has long been associated with recreation. Games of chance and strategy have been a part of human entertainment for centuries. People are captivated by the thrill of winning, so games and tricks with numbers have played an important part in people’s social lives across cultures. As mathematician and magician Gardner states, “…the delights of mathematical magic are greatest for those who enjoy both conjuring and mathematical recreations” (Gardner, 1956, p. xi). On pages 1-5 in this paper I will give a brief cross-cultural history of games using dice and playing cards and how they relate to recreational mathematics. Next, I will look at probability as it relates to games of chance. Beginning on page 5, I will describe a special card trick created by late mathematician and magician, William Fitch Cheney Jr., the recipient of the first Ph. D. in mathematics ever awarded at MIT in 1927. Finally, I will explain some of the mathematical concepts behind Cheney’s amazing Five Card Trick.

History of dice and playing cards

Games that the earliest people played probably involved drawing lots or guessing if a quantity was odd or even. These kinds of games of chance could be played with almost any material easily found at hand like seeds, pebbles, or shells and were played by people all over the world in different variations. It was common for ancient people to use divination practices to tell the future or to read signs to determine auspicious dates for planting, harvest and other important celebrations. The games we play today with playing cards and dice have been handed down and adapted through the generations from early divination practices with arrows. “Our present games are the survivals of these magical processes” (Hargrave, 1966. p.1). For as long as games of chance have held people captive with their allure of quick gain, these same games have been tied to the art of cheating. While playing games of chance, people emancipated themselves “…from
the laws of blind fate by means of a little dexterous management, vulgarly called cheating” (Chatto, 2002, p. 10). It is no wonder that common games of chance later became the raw material for stage magicians and parlor magicians as they used common objects and sleight of hand and illusion to perform their tricks.

**Dice games**

Knucklebones have been found at archeological sites all over the world. It is commonly believed that these bones were used in games of chance. They were sometimes made from animal or human bones and later they were made from clay. Knucklebones themselves might have originated with sheep or goat ankle bones, which had four long sides and two ends which were smaller and convex. The knucklebones did not usually land on the convex sides, but each throw had four different possible outcomes. One interesting origin story about the games of chance of knucklebones comes from the Greek historian Herodotus (ca. 430 BC). He wrote that the games were invented by the Lydians during a time of terrible famine when food had to be rationed such that people could eat only every other day. The games of chance kept them occupied on the days that they did not eat, but at the same time required very little energy. (Craig, 2002, p. 111)

Some of the world’s oldest board games involved rolling dice of some sort and counting. The British Museum has a board game from around 2600 BC called the Royal Game of Ur. It was played by rolling 3 sided pyramid shaped dice and moving through a series of boxes in a certain order to get the game pieces through the board and off again. There have been many similar board games found in different places. Some use tetrahedron dice while others used flat stick dice. (The British Museum has an interactive page where you can play a version of the Royal Game of Ur at [www.mesopotamia.co.uk/tombs/challenge/cha_set.html](http://www.mesopotamia.co.uk/tombs/challenge/cha_set.html).)
Perhaps one of the most commonly known games of chance and strategy from ancient times is mancala. The name mancala comes from an Arabic word which means ‘transferring’. This game has been played in different variations all over the world and is also known as Wari and Ayo in East Africa, and Sungka and Dakon in Asia. Game boards vary from beautifully carved and decorated sets, to holes scooped out of the dirt with a stick. Basically stones are transferred in turn from one of the holes to the others until one side of the board is cleared and the stones are counted in the end pots. This game is still enjoyed by people of all ages around the globe.

Abbia is another ancient dice game that was played by the Beti people of the rainforest of central Africa. The dice or chips used to play the game were from a certain tree called Mimusops Congolensis. The tree only gives fruit every other year and it is poisonous. The chips are made from the almond shaped pits of the fruits and are carved in relief so that the design is the dark outer part and the background is lighter colored. It is thought that the stakes for the game were very high because the game pieces themselves were so elaborately carved. Each player had his own set of chips, and traditionally everyone who played carved their own. The game was based on chance alone, not skill. Sometimes even women were lost in a toss.

Some ancient games with dice involved little strategy. For example, Nigbé, which was played on the west coast of Africa, was a game of chance using just 4 cowrie shells. In the game one person tossed the shells and was awarded points depending on how they fell. The luckiest roll was all four landing open side down, which earned 10 points. All four shells landing open side up was awarded 5 points. Two up and two down scored 2 points, while combinations of 3 and 1 were not awarded any points (Wilkins, 2002, p.23).
Wadi was played in central Africa by the Baongo people. Each player brought 8 two colored disks to play. Each player would place a bet and all would toss their own counters at the same time. Those whose disks landed with an even number of white sides up lost, while winners had an odd number of white sides up. The stakes were shared out among all the winners, with higher odd numbers winning a larger proportion of the pot. If there were no winners in a round, the pot would remain and the stakes for the next round would be added to it.

Several ancient games involved counting multiples of four. Panda was played in Africa and a similar game Witcli was played in North America. In both games player 1 tossed a handful from a basket holding a large number of objects such as shells or seeds. Expert players might use up to 300 shells, while beginners might start with about 40. Player 2 would then tell player 1 to add 1, 2, or 3 more shells to those on the ground. Then, the shells on the ground were counted off by fours. If there were exactly a multiple of four shells, player 2 wins.

In Pakistan, Pittar Pradash was a betting game played with 16 cowrie shells. Players each chose the number they think will win between 1 and 16 and place a marker on the number and their wager in the pot. One person shakes and tosses the 16 shells. The number of shells that land open side up is the winner. The winner gets the stakes and becomes the caster for the next round. If no one chooses the winning number, the stakes grow for the next round.

The place keeping game of Nigeria sounds a lot like what would later become a parlor magic trick performed with playing cards. Sixteen distinguishable stones were arranged into two lines of eight. Person 1 left the area to be out of earshot of the other people who would agree on which of the sixteen stones would be chosen as the secret stone. Person 1 was called back and asked the others in which row the secret stone was located. Keeping that row in mind, Person 1 rearranged the stones and asked a second time in which row the secret stone was located. This
was repeated two more times giving Person 1 four chances to ask the location of the secret stone and then rearrange the rows. After four turns, Person 1 would guess which of the sixteen stones the secret stone was.

“Dice games have been played in the southwest United States for more than 2000 years” (Craig, 2002, p.213). Gambling was common not just among the men, but the women and children as well. Games of chance were the most common form of game playing among Native Americans. “The aspect of associating gambling games with a purpose of increasing a tribes’ overall material wealth, particularly as it related to agriculture, is a theme often repeated among native North American tribes” (Craig, p. 144).

Two interesting games were played by the Omaha, Plum stone and Ja chawa. In the version of Plum stone played by the Omaha women, five plum pits were tossed into the air and caught with a basket. The plum pits were marked on one side and unmarked on the other. Some combinations of marked and unmarked sides up were awarded points. Versions of this game played with seven or nine stones were played by other tribes as well. The game of Ja chawa starts with a pile of sticks in the center. Each player on his or her turn has to take some, without counting them first. Play continues until all the sticks are gone. The winner is the person with the smallest odd number of sticks at the end.

Playing Cards

The popularity of dice games began to wane among people who were more interested in playing cards. Playing cards may have developed from dominos. Some of the oldest cards still in existence came from China in 969. (Wilkins, 2002, p. 61). Cards were used for fortune-telling and games of chance. Early Chinese cards were called money cards for their symbols of money they displayed on one side. A deck consisted of four suits numbered one through nine and an
extra wild card. The earliest examples were small, 1 inch by 3 inches, much like modern
dominoes, but thinner. Korean cards were about seven inches long and just half an inch wide.
Sometimes cards were elaborately decorated and depicted royal courts or other actual people.
Other cards were used for teaching and had proverbs or poems to be matched in pairs.

Playing cards probably got to Europe in the middle of the 14th century from Asia. It is
not clear exactly how that happened, but through travel and trade, the cards got to Europe and
flourished there. Some technological advances of the time helped playing cards become popular.
Early sets in Asia were hand painted, which was very time-consuming to make. Some cards
were made with woodcuts which were also time-consuming to make, but once the designs were
made, they could be used to produce multiple sets. Once the printing press was in place, it
became easier and less expensive to mass produce packs of cards, and their popularity spread
rapidly. The suits commonly used in Europe in the fifteenth century probably evolved from
Islamic cards.

Like dice in previous times, cards were used for games of chance or to make predictions,
and also to play games of skill. People of all classes enjoyed the games, and at different times,
many were admonished about the dangers or pitfalls of playing too much. In some European
courts, games of cards and gambling were forbidden to different groups of people including
soldiers, royal attendants, and women.

Many of the early European card games played are much like those that are still popular
today. One game from Korea, Yet-pang-mang-i, was a wagering game to make a combination of
19. Each player was dealt one card and could draw either one or two more. The sums were
added. If no one had exactly 19 the highest combination without going over 19 was declared the
winner. If two players had the same sum, the suit of higher rank would win (Wilkins, 2002, p. 71).

Different suits and ranks were developed in different countries and the cards we have today are related to that development. Eventually, fashionable families hired gaming masters to teach young ladies the basics of card playing much like they hired French and music tutors. This wide spread practice of playing cards was seen by some as a sign of society’s decline. The excess as seen in this quote from a London newspaper in 1753, “However absurd such a conduct in [P]arents may appear to the [S]erious and [S]ober minded, it is undeniably true that such a practice is now introduced by some, and will, it is feared, be adapted by many more” (as cited in Hargrave, 1966, p. 206). It is interesting to note that this sentiment is timely today if the subject were changed from card playing to texting, or surfing the web, or video games, or some other “new” pastime.

**Games of Chance and Probability Theory**

Italian mathematician, Girolamo Cardano (1501-1576), wrote the first treatment of the theory of probability after studying games of chance with dice, cards, and chess. He discovered that “Chance follows certain rules” (McElroy, 2005 p. 56). While his definition of probability is no longer used because he only dealt with equally likely outcomes, his work would give later French mathematicians Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662) a beginning place for their correspondence which eventually brought forth the branch of mathematics we now call probability.

According to Pappas, “Dice can be considered responsible for getting Blaise Pascal and Pierre de Fermat to focus their attention on probability” (Pappas, 1994, p. 134). In 1654 Pascal and Fermat considered two questions in letters to each other that would lead them to the
beginning of a new branch of mathematics. One question was about calculating the probability of rolling a pair of sixes in 24 rolls. The second question related to equitably dividing the stakes in a game of chance that is interrupted before it is finished. One conclusion was that “reliable information could be obtained from uncertain events as long as a large number of trials could be performed and measured” (McElroy, 2005, p. 204).

**Using cards to perform magic tricks**

There are many kinds of magic tricks using cards. Some of the common forms are: self-working or mathematical tricks, and tricks involving skills like sleight of hand or special shuffling, and those that use special products, for example, a marked deck, or other special cards. In this paper, we will only look at tricks that use mathematics, not gimmicks. The late Martin Gardner was a noted mathematician, magician, and thinker who, among other accomplishments, wrote a column entitled “Mathematical Games” for the magazine *Scientific American* for over 25 years. Gardner cautioned people about mathematical magic, even though he wrote extensively about it. “Mathematicians are inclined to regard it as trivial play, magicians to dismiss it as dull magic” (Gardner, 1956, p. xi). Regardless of the warning, there is plenty of amusement to be found in different sorts of mathematical games, puzzles and tricks. Mathematical card tricks have likely been around since playing cards themselves, but the earliest ones preserved came from the nineteenth century. Gardner (Gardner, 1956) outlines five properties which make a deck of playing cards ideal for performing mathematical tricks: 1) They can be used as counters, 2) they are numbered, 3) they are divided into four sets (suits), and further divided by color, 4) they have different fronts and backs, and 5) they are portable. Some of the ways card tricks relate specifically to mathematics include combinatorics, counting, probability, and modular
arithmetic. Some self-working tricks are those based on counting, and require understanding and technique, but not necessarily the typical “magician” skills, like palming cards, false shuffling, or peeking. Many mathematical card tricks also require proficiency in those and other basic card handling skills.

**Fitch Cheney’s Five Card Trick**

The trick I am going to present is credited to mathematician William Fitch Cheney, Jr. (1894-1974). Cheney was a mathematician and professor for most of his life, but enjoyed performing magic tricks for family and friends. He was a frequent contributor to the magazine M-U-M, the official publication of the Society of American Magicians. The magical effect of the Five Card trick is that the magician can name a card drawn randomly from a deck of cards. As originally published in Lee’s (1950) book, *Math Miracles*, it was set up as a telephone trick. After a member of the audience selected five cards at random, one magician turned one card face down and laid the remaining four face up in a line on the table, and then phoned a second magician. Magician 1 would name the four cards faced up on the table and magician 2 would name the hidden card. In this way it was clear to all in the audience that there were no body language signs between the two magicians. In the version that will be examined here, the magician is simply out of the room. The assistant asks for a volunteer to draw five cards at random from a standard deck of 52 cards. The cards are drawn and given back to the assistant, who chooses one of the cards which will then be hidden. The assistant then lays the remaining four out on the table. The magician is called back, surveys the four cards on the table, and names the hidden card.

The probability of correctly naming 1 random card chosen from a standard deck of cards is 1 in 52. This means that we are looking for just one card, called the favorable outcome, in a
set of 52 cards, the total possible outcomes. The probability of naming the chosen card is 1 in 52 or about 2%. In this trick, since four additional cards are shown to the magician, the chance of naming the hidden card correctly is just slightly higher, because now the set of cards has been reduced from the total of 52 possible outcomes to 48. The probability of naming the hidden card is now 1 in 48, which is still approximately 2%. With that probability, there would be no magic trick. What do the magician and the assistant know in order to make this trick work? There must be more than what we can see, but there is no telepathy or secret body language involved, which is why it could originally be performed over the phone. The magic is in the mathematics behind the trick. It is essential that the assistant and magician work together in order to make the effect work.

Think about what the magician sees: there are four cards face up in a line on the table. Can these four cards give the magician a clue as to what the hidden card is? The cards are selected by a member of the audience, so the assistant does not get to choose them, but the assistant does have all cards before the magician sees them, and in fact chooses which of the cards will be hidden before laying out the other four. This is where the mathematics comes into play. There are four cards on the table and 48 possible cards that could be hidden. How can the assistant and magician use the four cards to identify the one that is hidden if every time the trick is performed different cards are chosen? The solution to the trick has three mathematical components: the pigeonhole principle, modular arithmetic, and permutations. Using these three principles and following some conventions previously practiced will make the most of the magical effect. We will look at how each one of these concepts works in the Five Card Trick.
The Pigeonhole Principle

The pigeonhole principle is a simple, but important part of combinatorics. Let $n =$ the number of pigeons and $k =$ the number of pigeonholes. We define the ceiling, shown as $\left\lceil \frac{n}{k} \right\rceil$, to be the smallest integer $\geq \frac{n}{k}$. The floor of $\frac{n}{k}$, shown as $\left\lfloor \frac{n}{k} \right\rfloor$, is the greatest integer $\leq \frac{n}{k}$. When $n$ pigeons are put into $k$ pigeonholes, there exists at least 1 pigeonhole containing not less than $\left\lceil \frac{n}{k} \right\rceil$ pigeons and at least 1 pigeonhole containing not more than $\left\lfloor \frac{n}{k} \right\rfloor$ pigeons. For example, let $k = 4$, to represent the four suits in a deck of cards (the pigeonholes), and let $n = 5$ to represent the number of cards a volunteer draws from the deck for the trick (the pigeons). We know that $\frac{5}{4}$ is $1 \frac{1}{4}$, so the ceiling is 2. Therefore, at least one suit (a pigeonhole) has more than one representative among the 5 cards selected (the pigeons). Simply stated this means that if you draw any five cards, at least 2 of them will match suit.

To relate this to our trick we think of the 5 cards selected by a volunteer and the four possible suits. The pigeonhole principle guarantees that in any given draw of five cards, at least two of the cards are of the same suit. It is possible that less than all of the four suits are represented in the five cards selected, but we know that at least two of the cards must have the same suit. This is the first part of the solution.

One of the cards that share the same suit will become the hidden card and one will become the key card in the trick. The assistant scans the five cards to find a pair of cards of the same suit. The assistant must then quickly decide which of these two will be the hidden card and which will be the key card that the magician will see on the table with the other three cards. The assistant then shows the card that will be hidden to the audience and turns it face down on the
table. The key card is then placed in the first position face up on the table. Scanning left to right, it will be the first of the four cards the magician is allowed to see before naming the hidden card. The magician will know immediately the suit of the hidden card.

**Modular Arithmetic**

The assistant must also know how to count the numbers or faces of the chosen cards to find the difference between the value of the hidden card and the key card. This is done by using modular arithmetic. Modular arithmetic is a system that lets us think about operations of numbers using repetitive cycles outside the usual base ten system. If we want to add numbers using only those available to us in one suit of our standard deck of cards, we can use \((\text{mod } 13)\). In each suit the numbered cards are given their corresponding value, and then we assign the value of one to the ace, 11 to the jack, 12 to the queen, and 13 to the king to make our system of thirteen complete. In base 10 we can easily add 4 and 6 to get ten, and it is the same \((\text{mod } 13)\), but what happens when we want to add 10 and 6? We know \(10 + 6 = 16\), but in our deck of cards, we do not have a 16 card, we can only count as high as the number 13 since the suit has only 13 cards. Using \((\text{mod } 13)\), numbers higher than 13 will be expressed by the congruence of their remainders in \((\text{mod } 13)\). Let’s imagine one of the suits from our deck arranged in a circle, like the face of a clock (see Figure 1).
Using (mod 13) we start counting clockwise with the ace (1), 2, 3, …, 11 (jack), 12 (queen), 13 (king), and then we repeat the cycle. So, instead of saying 14, as we usually would, we start again with 1. We count 12, 13, 1, 2, …, and so on. Therefore, if we add 10 and 6, we count 11 (jack), 12 (queen), 13 (king), 1, 2, and then 3. We can add from any number in the circle and when we reach 13, we go around again. The base ten equation $10 + 6 = 16$ can become a new problem in (mod 13). When we divide 16 by 13 we get a quotient of 1 with a remainder
of 3. This remainder becomes our congruency answer in (mod 13). Using symbolic notation for (mod 13) we can show that \(10 + 6 \equiv 3 \pmod{13}\). So, 10 add 6, counting clockwise, is 3.

Let’s try another example. This time let’s start with the queen (12) and add 2. We know in base ten 12 and 2 are 14. Looking at the clock arrangement of the suit, we can count clockwise from the queen 2 places and land on the 1 (ace). In (mod 13), we can say that \(14 \equiv 1\) (mod 13). This means that when we divide 14 by 13 there is a remainder of 1.

This clock arrangement is a very quick and convenient way for the assistant in the trick to study the five cards the audience member has chosen and decide what to do next. Seen in this repetitive cycle any two cards are no more than six positions apart moving clockwise. These six spaces are important to the trick, and we will see why in the next section. How do the assistant and the magician use modular arithmetic to communicate the value of the hidden card? We have already established that the suit of the key card (the first upturned card) is the same as the suit of the hidden card. That narrows down the number of cards that could be the hidden card from 48 to 12; the remaining cards in the suit of the key card. When scanning the cards, the assistant chooses a pair of cards with the same suit. The assistant chooses which of the pair will be hidden and which will be the key card based on the difference in the values which cannot exceed six, for reasons we will see in the next section. For example, if the pair is the 2♥ and 8♥, the 8♥ will be hidden and the 2♥ becomes the key card, since \(2 + 6 \equiv 8 \pmod{13}\). The two cards cannot be switched, because to go clockwise from 8♥ to 2♥ the difference is 7 positions on the clock. Or we could say \(8 + 7 \equiv 2 \pmod{13}\). The congruence is true in (mod 13), but this model will not work in our five card trick, where we must limit the differences to 6 positions.

In another example, let’s say the pair is 4♥ and Q♥. We need to find the arrangement of these two cards with a difference \(\leq 6\) positions. Counting clockwise from 4♥ to Q♥ is 8 positions,
4 + 8 ≡ 12 (mod 13), but from Q♥ to 4♥ is only 5 positions, 12 + 5 ≡ 4 (mod 13), so the 4♥ is hidden and the Q♥ becomes the key card. With the key card Q♥ on the table as the first card faced up, how can the assistant use the remaining three cards to let the magician know which heart is hidden? And why must the differences in their positions be 6 or less? For the Cheney Five Card Trick, we use a code based on permutations.

**Permutations**

Recall that the volunteer has drawn five cards at random and handed them to the magician’s assistant, while the magician is out of the room. The assistant looks at the cards, decides which one will be hidden, shows it to the audience, and lays it face down on the table. The assistant has also identified the key card, which matches the suit of the hidden card, and has used (mod 13) to calculate the difference in positions on the clock of the two cards, keeping the difference of 6 or less positions moving clockwise between the key card and the hidden card. The three cards that the assistant still holds will all be distinct, but there is no way of knowing until the trick is being performed which specific cards from the deck they will be. The probability of simply guessing the hidden card now would be 1/12, an improvement over 1 in 48 as at the beginning, but there are still too many ways to go wrong to have a good magic trick. The third part of the solution involves using the possible permutations of the three cards to formulate a code which can transmit the value of the hidden card. The code is based on the arrangement of the three cards.

A permutation is an ordered arrangement of objects. If we have a collection of three objects, the permutations of three will show all the different ways those three things can be arranged. Let’s say we have three cards J♣, Q♦, K♣ as shown in Figure 2.
How many different ways can we arrange them? These are the permutations of the three cards:

1) J♠, Q♦, K♣ 
2) J♠, K♣, Q♦ 
3) Q♦, J♠, K♣ 
4) Q♦, K♣, J♠ 
5) K♣, J♠, Q♦ 
6) K♣, Q♦, J♠ 

We can use factorial notation 3!, which means 3 x 2 x 1, to show that there are 6 ways to arrange 3 objects.

The magician and the assistant must have an order worked out ahead of time, so that the assistant can arrange the cards into one of the six possible permutations and the magician will decode the permutation to find the hidden card. Since there are only 6 possible arrangements of the three cards, the addend discussed in the previous section must be less than or equal to 6. This means that with any given pair, the hidden card can be a maximum of 6 positions from the key card when counting clockwise on the clock model. In this trick the suits are arranged in alphabetical order, such that clubs are the lowest, followed by diamonds, then hearts, and finally the highest suit is spades (see Figure 3). Within each suit, the cards are arranged with aces low.

The aces are assigned the value of 1, number cards their corresponding value, jacks 11, queens 12, and kings 13. The order of the entire deck is determined first by suit and then by rank.
this order, the lowest card of the deck is the A♦ and the highest is K♠. For example, look at the following five cards: 2♠, 4♦, 4♥, 8♦, and K♣. Their arrangement from low to high following the order of suits for the trick would be: K♣, 4♦, 8♦, 4♥, 2♠.

No matter which three cards the magician’s assistant has after choosing the hidden card and laying down the key card, the cards can be put in order as low, medium, or high. This order, or permutation, can represent a code for an addend. The code will tell the magician how many spaces on the clock to add to the key card, which is in the first position face up on the table, to get to the hidden card.

Using the permutations the 3 cards can be arranged in 3! = 6 distinct orders, as shown in the code below.

1 = low, middle, high
2 = low, high, middle
3 = middle, low, high
4 = middle, high, low
5 = high, low, middle
6 = high, middle, low.

Let’s work out a few examples. The assistant gets the following cards from the volunteer: 3♣, 7♣, 10♣, 7♥, and K♠ (see Figure 4). The assistant chooses the 2 cards of the same
suit which are the 3 and 7 of clubs. The assistant decides that the 7♣ will be the hidden card since $3 + 4 \equiv 7 \pmod{13}$. Only this order will work since $7 + 9 \equiv 3 \pmod{13}$ requires an addend of 9 and $9 > 6$. The 3♣, the key card, is laid down in the first position on the table and the remaining three cards are arranged according to the order of suits corresponding to the permutation middle, high, low, the code to show that 4 should be added to the 3♣, to get to the hidden 7♣ using the clock model previously discussed.

Figure 4
Hidden 7♣

![This is the key card.]

![This is the hidden card.]

Now let’s try an example going from a card with a higher value to a card with a lower one. This time let’s say that the cards the volunteer selects are: 5♣, A♦, 8♣, 4♠, and J♠ (see Figure 5). The two cards of matching suit are the 4♠ and J♠. The assistant must hide the 4♠ and make the J♠ the key card. Using the clock arrangement (see Figure 1), and counting clockwise, we need to add six to the J♠ to get to 4♠. In (mod 13) we show this as $11 \text{ (jack)} + 6 \equiv 4 \pmod{13}$. The assistant needs to select the hidden card with care since moving from 4♠ to J♠ on the clock
would require adding seven positions, $4 + 7 \equiv 11 \text{(jack)} \pmod{13}$. (With only the six permutations of 3 objects for the code, this arrangement does not work for the trick.) Therefore, the remaining three cards, $5\spadesuit$, $8\spadesuit$, $A\diamondsuit$, need to show “add 6”, and so are arranged in high, middle, low ($A\diamondsuit, 8\spadesuit, 5\spadesuit$) order according to the permutation code.

![Figure 5](image)

**Figure 5**

$J\spadesuit$ add 6 to $4\spadesuit$

This is the key card.

This is the hidden card.

**Variations**

After performing the trick once or twice, it might become obvious that the key card in the first position on the table is the clue to the suit of the hidden card. One way to vary the position of the key card is to use modular arithmetic in a new way. Look at the set of cards as shown in Figure 6: $5\spadesuit$, $A\spadesuit$, $9\heartsuit$, $4\spadesuit$ and $J\spadesuit$. The $4\spadesuit$ is hidden, so we find the sum of the remaining four cards giving the ace the value of 1 and the face cards the values of jack = 11, queen = 12, and king = 13. In this case the sum of the four face up cards is 26. Convert the sum to its congruent value in $(\text{mod } 4)$, $26 \equiv 2 \pmod{4}$ and place the key card, $J\spadesuit$, in the second position, which
corresponds to the congruency statement. The remaining 3 cards are put in the order of high, middle, low, around the key card for the permutation code indicating add 6. The magician comes in, quickly finds the sum of the four cards, determines the position of the key card, looks at the arrangement, and correctly announces the hidden 4♠. This variation would only require a bit more preparation and practice, but it would make the trick much more difficult for a spectator to figure out after seeing it just a couple of times.

**Figure 6**
Key card (mod 4)

Now, let’s imagine another variation to make the trick appear even more difficult for the magician. This time, let’s change the code from the arrangement of the three cards from low, middle, and high, to face up or face down. A good bit of drama could be added here for effect. Can the magician guess the missing card with only three cards to see? What about two? Suppose the volunteer hands the assistant the following cards: K♣, 2♦, 4♥, 3♠, and 5♠. The cards of matching suit are the 3♠ and 5♠, so the 5♠ is hidden. The 3♠ is the key card and the remaining 3
cards need to show the magician to add 2 to the 3♠ to get to 5♠ using the clock model. Using a different code for the six permutations of the three remaining cards showing them face up (U) or face down (D), we can now present the six addends this way:

1 = U D D
2 = D U D
3 = D D U
4 = D U U
5 = U D U
6 = U U D

In this case with the key card, 3♠, and the hidden card, 5♠, we want to show add 2, so we arrange the cards D U D (see Figure 7) after the key card.

There are even more variations of the trick. In one variation called Fitch Four Glory, four cards are selected and only three are displayed. To make this trick work, the suits need to be redefined from the usual four suits of 13 to just three suits of 17 each. These new suits are called alpha, beta, and gamma. Alpha is the entire suit of 13 clubs, A♣ through K♣, with the following
4 spades added after the K♣ to complete the suit of 17: 2♠ (14), 3♠ (15), 4♠ (16), and 5♠ (17).

Beta is the entire suit of 13 diamonds with the next four spades added after the K♦ to complete the suit of 17: 6♠, 7♠, 8♠, and 9♠. The third suit, gamma, is the entire suit of 13 hearts with the next four spades added after the K♥ to complete the third suit of 17: 10♠, J♠, Q♠, and K♠. This leaves three “super” suits with 17 cards each and the A♠ left over. With these three new suits, we could use (mod 17) in the clock arrangement like we did with the original trick in (mod 13). In this variation there would be differences of up to eight positions between any two cards.

Therefore, the code must be modified to find three new arrangements. The “super” suits are ordered low to high starting with alpha: A♣ through K♣, then 2♠, 3♠, 4♠, and 5♠; followed by beta: A♦ through K♦, 6♠, 7♠, 8♠, and 9♠; and finally gamma: A♥ through K♥, 10♠, J♠, Q♠, and K♠. Since A♠ is not part of the “super” suits, it does not follow this order. Combining the low, middle, high permutations and the face up and face down code, the addend 7 could be made by keeping the three cards face up and placing the key card in the first position, followed by the lower of the two remaining cards, and finally the higher of the two. The addend 8 could be made by arranging the three cards face up with the key card first, followed by the higher of the two remaining cards and finally the lower of the two. The complete code for the Fitch Four Glory variation would be:

1 = U D D (key card in position 1)  
2 = D U D (key card in position 2)  
3 = D D U (key card in position 3)  
4 = D U U (key card in position 2)  
5 = U D U (key card in position 1)  
6 = U U D (key card in position 1)  
7 = U key card, U lower, U higher  
8 = U key card, U higher U lower  
A♠ = D D D
As seen in the code, each of the other arrangements 1-6 would include at least one card face up, so the first face up card would be the key card and would communicate the suit of the hidden card. If the hidden card is the A♠, which was excluded from our three super suits, the assistant would lay the remaining three cards face down and the magician would guess the A♠.

**Conclusion**

The Cheney Five Card Trick is a prime example of using mathematics for recreation in a magical application. The Five Card Trick, like the best mathematical magic, “combines the beauty of mathematical structure with the entertainment value of a trick” (Gardner, 1956, p. xi). We have seen that throughout history people have always enjoyed using math in games with dice and cards. People began by playing games of chance with just a few simple materials and concepts. Over time, our games have become more complex and elaborate, but the mathematical concepts, the thrill of chance, and the allure of the unknown has remained the same. When mathematics, magic, and mystery are combined like this we can truly experience the joy of mathematics.
References


