

Powers

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In partial fulfillment of the requirements for the Master of Arts in Teaching with a
Specialization in the Teaching of Middle Level Mathematics in the Department of
Mathematics.

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July 2009

INTRODUCTION

Powers, or exponents, give us a convenient, shorthand way to denote that a number is multiplied by itself a given number of times. The number being multiplied is called the base, and the number of times it appears in the product is the power.

READING AND WRITING POWERS

The use of powers, or exponents, is a mathematical operation of repeated multiplication involving two components. The first part is the base, and the second part is the exponent, which was at one time referred to as the index. The exponent is written to the right of the base in the super-script.

base $\rightarrow 4^8$ \leftarrow exponent

This notation is read as “the base raised to the power of the exponent”. The example above would read “4 to the 8th power”. This could also be read as simply “4 to the 8th”. Some exponents have their own special name, such as the two in b^2 , which is read “b squared”, and the three in b^3 , which is read “b cubed”.

So, the symbol 4^2 is read as “4 squared” which means $4*4$, and the symbol 4^3 is read as “4 cubed” which means $4*4*4$.

The history of how exponents and powers are read and written dates back to the 1570’s. Euclid (approx 330-275 B.C.) first used the term ‘power’ for the square of a line. It is assumed, however, that Euclid was thinking of the geometric square and not of a new number formed by multiplying the number by itself. In 1696, Samuel Jeake, in his book *Arithmetic*, was the first to use the word ‘index’: Jeake writes, “Mark their indices or how many degrees the Number you would produce is removed from the Root as whether it be second, third, fourth, etc”.

The superscript notation was first written in the latter part of the 15th century and published in 1880 by Chuquet in Triparty. He wrote 5 , 5^1 , 5^2 , 5^3 where we would write 5 , $5x$, $5x^2$, $5x^3$.

EXPONENTS INDICATE A REPETITION OF MULTIPLICATION

Exponents are learned as a way to express repeated multiplication. The base indicates the number being multiplied, and the exponent indicates the number of times the base is to be multiplied by itself. For example, 4^2 denotes the number arrived at by taking the base, 4, and multiplying it by itself two times; so, $4^2 = 4*4 = 16$.

EXPONENTIAL RULES

A definition for basic exponent rules can be established using examples. The exponential rules are given using a non-zero real number b (for the base) and positive integers m and n (used as exponents). After these rules are established, we extend them to using rational and negative exponents. The first exponent rule to establish is: $b^m b^n = b^{m+n}$

$$b^m b^n = \underbrace{b * b * b * b \dots}_{\text{“m” factors of b}} * \underbrace{b * b * b * b \dots}_{\text{“n” factors of b}}$$



“m” factors of b; “n” factors of b

Since the total number of factors in the new product is $m+n$, it is clear that $b^m b^n = b^{m+n}$

To show how this exponent rule works with specific values, let $b = 2$, $m = 3$, $n = 4$. We have

$$2^3 2^4 = (2 * 2 * 2) * (2 * 2 * 2 * 2) = 2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7 = 2^{3+4}$$

The next exponent rule established through example is $(b^n)^m = b^{mn}$.

$$(b^n)^m = \underbrace{(b^n) (b^n) (b^n) (b^n)}_{\text{“m” times}}$$



“m” times

To show how this exponent rule works with specific values for exponents, let $n = 2$ and $m = 4$

$$\begin{aligned} (b^2)^4 &= (b^2)(b^2)(b^2)(b^2) \\ &= (bb)(bb)(bb)(bb) \\ &= bbbbbb \\ &= b^8 \\ &= b^{2*4} \end{aligned}$$

To show how this generalization works with specific values for the base and exponent, let $b = 2$, $n = 3$, $m = 4$. Then

$$\begin{aligned} (2^3)^4 &= (2^3)(2^3)(2^3)(2^3) \\ &= (2 * 2 * 2)(2 * 2 * 2)(2 * 2 * 2)(2 * 2 * 2) \\ &= 2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 \\ &= 2^{12} \\ &= 2^{(3)(4)} \end{aligned}$$

The establishment of these rules, by example, allows us to extend these rules to apply to situations involving more complex exponents.

EXTENSION OF EXPONENTIAL RULES

At this point, the exponent rules that establish that $(b^n)^m$ is the same as b^{mn} extends to the case where the exponents are rational numbers. So, let $b > 0$ and consider b^n , which is equal to $b^{1/n}$. When we see this symbol, we must ask ourselves: “*what number, when raised to the n th power, results in b ?*” We consider $7^{1/3}$ to explore this concept, Realizing that we can’t simply multiply 7 by itself $1/3$ times to find the answer,

we rely on and extend the property established above, namely that $(b^n)^m$ is the same as b^{mn} . So, consider $(7^{1/3})^3$, which is the same as: $7^{1/3}$ times $7^{1/3}$ times $7^{1/3}$. $(7^{1/3})^3$, then, can also be thought of as $7^{3/3}$, which is 7. We arrive at the following definition: $7^{1/n}$ is precisely that positive number such that $(7^{1/n})^n = 7$.

Consider $b^{p/n}$, where p and n are positive integers, which can also be written as $(b^p)^{1/n}$. The following can be said: $b^{1/n}$ raised to the p power, n times, is equal to

$$\underbrace{(b^{1/n}) (b^{1/n}) (b^{1/n}) (b^{1/n}) \dots}_{p \text{ times}} \quad \underbrace{(b^{1/n}) (b^{1/n}) (b^{1/n}) (b^{1/n}) \dots}_{p \text{ times}}$$

$$\underbrace{\hspace{10em}}_{n \text{ times}}$$

So, it follows that this equals b^p

Note: we have defined $b^{p/n}$ to be precisely that number such that when raised to the n th power, the result is b^p . We have shown here that using the exponent rules developed up to here in this paper, it is in fact true that $(b^{p/n})^n = b^p$.

DETERMINING THE VALUE OF b^0

In the case of b^0 , I began with wondering what b *should* equal in this case. I can go back to my previous definitions ($b^m b^n = b^{m+n}$), and think about how that would look if $m = 0$ or if $n = 0$. Consider the product $b^0 b^3$. Based on my established definitions, this must equal b^{0+3} , which simplifies to b^3 . Since we have shown that $b^0 b^3 = b^3$ for all numbers b , then b^0 must be the number 1. So, we define $b^0 = 1$. I will check this definition using my already established definitions:

Consider $(b^m)^n = b^{mn}$. If either $m = 0$ or $n = 0$, the right hand side is $b^0 = 1$. Likewise, on the left hand side we have $(b^0)^n = 1^n = 1$ if $m = 0$, and $(b^m)^0 = 1$.

NEGATIVE EXPONENTS

Often, when negative exponents are explored, they are viewed as a pattern, such as $2^2 = 4$, $2^1 = 2$, $2^0 = 1$ and $2^{-1} = 1/2$. This pattern, though true, does not connect negative exponents to the rules established above. To establish the use of negative exponents using the rules established, we will first consider the exponent rule $b^m b^n = b^{(m+n)}$ together with the negative exponential b^{-n} , where n is a positive integer. Consider $b^n b^{-n}$; I want this expression to equal $b^{n+(-n)} = b^0 = 1$. So, to make the equation $b^n b^{-n} = 1$ true, we must create the following definition: $b^{-n} = 1/b^n$. I'll verify that this definition is consistent with the other established definitions above. Consider the equation $b^m b^n =$

b^{m+n} —I'll substitute values for the exponents. Let $m = -2$ and $n = 7$. I want $b^{-2}b^7$ to be equal to b^{7-2} , which is equal to b^5 .

Using our definition, we simplify $b^{-2} b^7 = 1/b^2 (b^7) = b^7/b^2 = b^5$, as I hoped.

For further verification:

I want $b^{-m}b^n$ to be equal to b^{-m+n} .

Using our definition, we have that $b^{-m}b^n = 1/b^m b^n = b^n/b^m$

If $n > m$, this is really $n-m$ copies of b multiplied together. If $m > n$, this is really $m-n$ copies of b divided into 1, which is $1/b^{m-n} = b^{-(m-n)} = b^{n-m} = b^n/b^m = b^{n-m}$.

Now, I must look to see if $(b^m)^n = b^{mn}$ holds true.

We look at $(b^{-m})^n$, where m is a positive integer:

$$(b^{-m})^n = (1/b^m)^n = 1/b^m 1/b^m \dots 1/b^m = 1/(b^m b^m \dots b^m) = 1/b^{mn} = (1/b)^{mn} = b^{-mn}.$$

In a similar manner, I can check this claim if $n = 0$ or if $m = 0$.

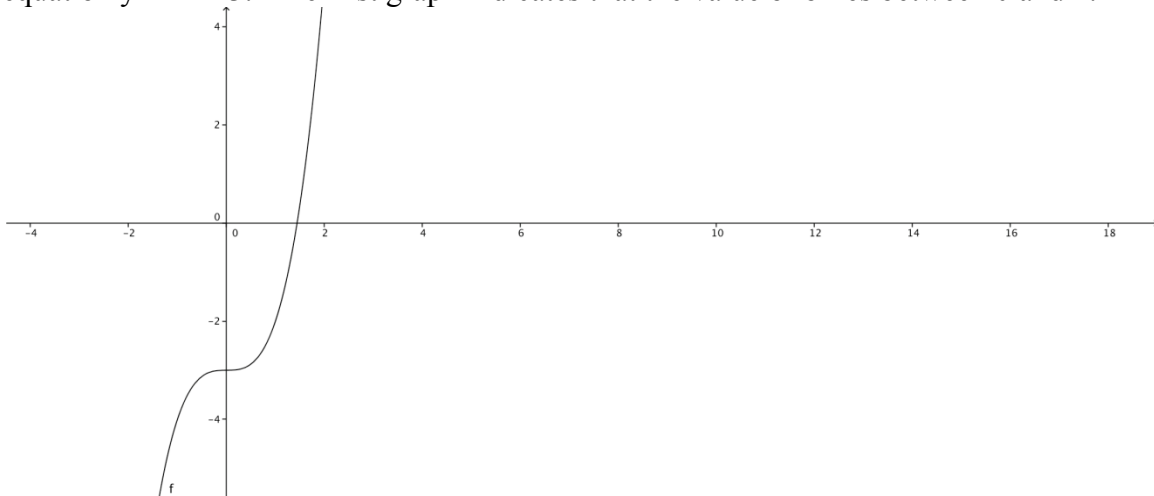
TWO WAYS TO GIVE AN APPROXIMATION FOR $3^{1/3}$, TO SIX-DIGIT ACCURACY

There are a variety of ways to give an approximation of values that are not apparent. The most common way to approximate values is by using a calculator. Using a calculator, one finds that a decimal approximation for $3^{1/3}$ is 1.44224957. Other methods require more time and labor to determine a decimal approximation and are thus less popular, but they achieve the same result. One way is to trap the value $3^{1/3}$ between two integers by looking at integer powers near its value. Another way is to explore the value through its graph on Geogebra.

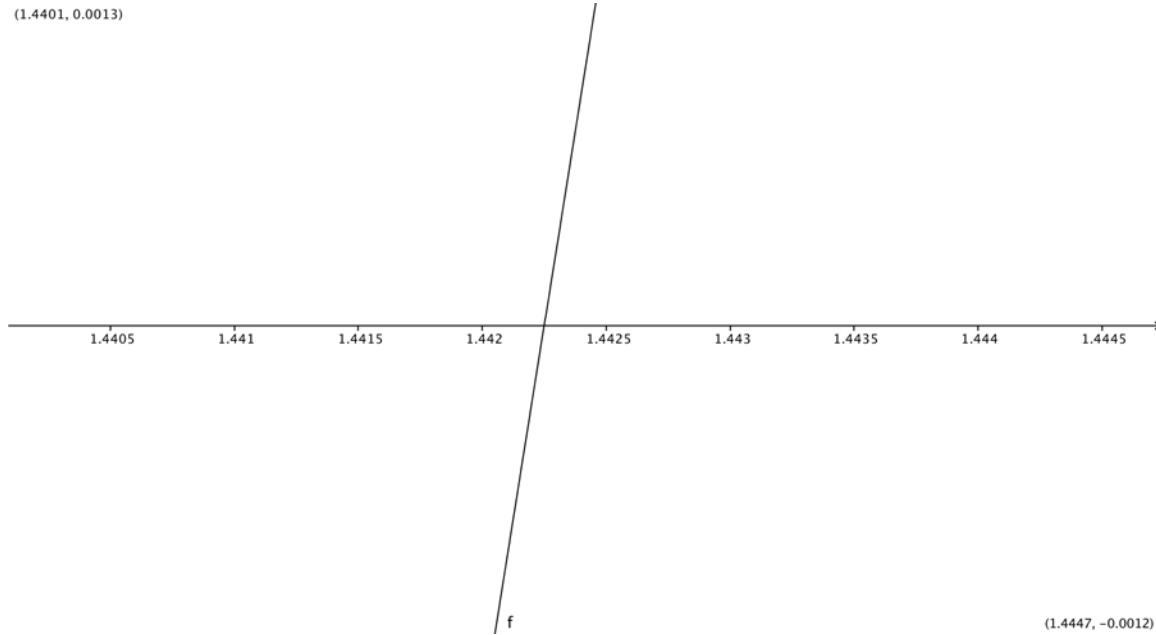
The goal here is to find the value b such that $b^3 = 3$. The calculator has a certain amount of accuracy, but the method I use below—using integers as a way to estimate the value of b —is an attempt to achieve even greater accuracy. I originally knew that the value of b is between 1 and 2, because 1^3 is 1 and 2^3 is 8. Knowing that 1 is too low of an estimation and 2 is too high of an estimation, I found the midpoint between 1 and 2 and continued to narrow the numerical gap of possible values for b . After calculation of $1.5^3 = 3.375$, I determined that 1.5 is still an overestimate for the value of b , so I then found the midpoint between 1 and 1.5. I continued this method, narrowing these values so that b was always trapped between two values which were too high and too low, respectively, until I could approximate the value of b to at least six decimal places.

Too Low	Too High	Estimate	(Estimate) ³
		1	1
		2	8
1	2	1.5	3.375
1	1.5	1.25	1.953125
1.25	1.5	1.375	2.599609375
1.375	1.5	1.4375	2.970458984
1.4375	1.5	1.46875	3.168426514
1.4375	1.46875	1.453125	3.068378448
1.4375	1.453125	1.4453125	3.019154072
1.4375	1.4453125	1.44140625	2.994740546
1.44140625	1.4453125	1.443359375	3.006930791
1.44140625	1.443359375	1.442382813	3.000831542
1.44140625	1.442382813	1.441894531	2.997785012
1.441894531	1.442382813	1.442138672	2.999308019
1.442138672	1.442382813	1.442260742	3.000069716
1.442138672	1.442260742	1.442199707	2.999688851
1.442199707	1.442260742	1.442230225	2.99987928
1.442230225	1.442260742	1.442245483	2.999974497
1.442245483	1.442260742	1.442253113	3.000022106
1.442245483	1.442253113	1.442249298	2.999998301
1.442249298	1.442253113	1.442251205	3.000010204
1.442249298	1.442251205	1.442250252	3.000004253
1.442249298	1.442250252	1.442249775	3.000001277
1.442249298	1.442249775	1.442249537	2.999999789
1.442249537	1.442249775	1.442249656	3.000000533

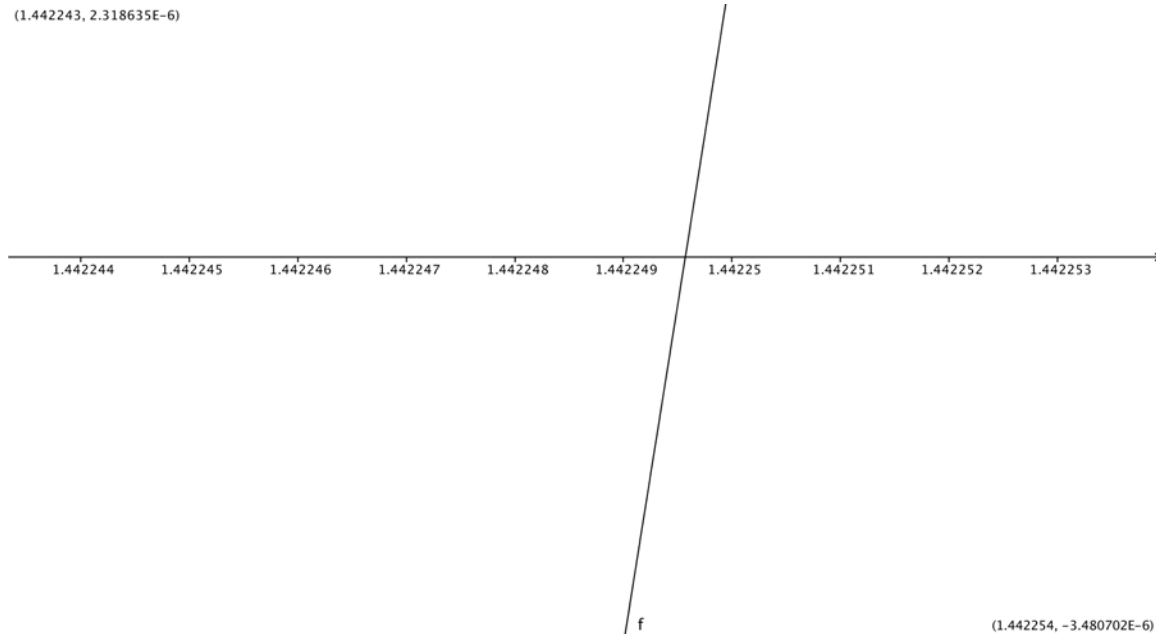
I also was able to verify this approximation using Geogebra. The graphs below show the equation $y = x^3 - 3$. The first graph indicates that the value of b lies between 0 and 2.



The next graph from Geogebra is a zoomed version showing that the value of b lies between 1.442 and 1.4425.



The final graph, which was the closest available from Geogebra, shows the value of b to lie between: 1.442249 and 1.44225.



The three methods used to approximate $3^{1/3}$ all yielded approximately the same value: The calculator's approximation was 1.44224957, the table's approximation was

1.442249656, and the graph on Geogebra showed the value to be between 1.44249 and 1.44225.

EXTENSION OF DEFINITION TO $b^{1/3}$, WHEN b IS IRRATIONAL

The method outlined above will work for any rational number. If the base is irrational, the approach is similar. The following examples show how we approximate $\sqrt{2}^{1/3}$ and $\pi^{1/3}$ up to six decimal places by using rational numbers as exponents to narrow the gap in which the actual values lie.

For $b = \sqrt{2}$, I can write $(\sqrt{2})^{1/3} = (2^{1/2})^{1/3}$. Using the rule established above, $(2^{1/2})^{1/3} = 2^{(1/2)(1/3)} = 2^{1/6}$. So, I'm asking what number, when multiplied to itself 6 times, yields 2—I want to solve $k^6 = 2$ for k . According to my calculator, the value for k is approximately 1.1222462048. In this table, my goal is to make my estimate, when taken to the 6th power, as close to 2 as possible.

Too Low	Too High	Estimate	Estimate ^ 6
		1	1
		2	64
1	2	1.5	11.390625
1	1.5	1.25	3.814697266
1	1.25	1.125	2.02728653
1	1.125	1.0625	1.438711226
1.0625	1.125	1.09375	1.712018275
1.09375	1.125	1.109375	1.864104472
1.109375	1.125	1.1171875	1.944269247
1.1171875	1.125	1.12109375	1.985416325
1.12109375	1.125	1.123046875	2.006260406
1.12109375	1.123046875	1.122070313	1.995815689
1.122070313	1.123046875	1.122558594	2.001032368
1.122070313	1.122558594	1.122314453	1.99842261
1.122314453	1.122558594	1.122436523	1.999727135
1.122436523	1.122558594	1.122497559	2.000379663
1.122436523	1.122497559	1.122467041	2.000053377
1.122436523	1.122467041	1.122451782	1.99989025
1.122451782	1.122467041	1.122459412	1.999971812
1.122459412	1.122467041	1.122463226	2.000012594
1.122459412	1.122463226	1.122461319	1.999992203
1.122461319	1.122463226	1.122462273	2.000002398
1.122461319	1.122462273	1.122461796	1.999997301
1.122461796	1.122462273	1.122462034	1.999999849
1.122462034	1.122462273	1.122462153	2.000001124
1.122462034	1.122462153	1.122462094	2.000000487

The values shown in my table verify the calculation performed on my calculator.

Similarly, when $b = \pi$, my goal is to find a number whose cube is within 6 decimal places of π . I'll use the same method of adjusting my estimates based on

whether they are too low or too high to narrow them in on the value of π . According to my calculator, $\pi^{1/3}$ is approximately 1.464591888.

Too Low	Too High	Estimate	(Estimate) ³
		1	1
		2	8
1	2	1.5	3.375
1	1.5	1.25	1.953125
1.25	1.5	1.375	2.599609375
1.375	1.5	1.4375	2.970458984
1.4375	1.5	1.46875	3.168426514
1.4375	1.46875	1.453125	3.068378448
1.453125	1.46875	1.4609375	3.118134975
1.4609375	1.46875	1.46484375	3.143213689
1.4609375	1.46484375	1.462890625	3.130657591
1.462890625	1.46484375	1.463867188	3.136931452
1.463867188	1.46484375	1.464355469	3.140071523
1.464355469	1.46484375	1.464599609	3.141642344
1.464355469	1.464599609	1.464477539	3.140856868
1.464477539	1.464599609	1.464538574	3.14124959
1.464538574	1.464599609	1.464569092	3.141445963
1.464569092	1.464599609	1.464584351	3.141544153
1.464584351	1.464599609	1.46459198	3.141593248
1.464584351	1.46459198	1.464588165	3.1415687
1.464588165	1.46459198	1.464590073	3.141580974
1.464590073	1.46459198	1.464591026	3.141587111
1.464591026	1.46459198	1.464591503	3.14159018
1.464591503	1.46459198	1.464591742	3.141591714
1.46459198	1.464591742	1.464591861	3.141592481

The value in the third column of my table, when the fourth column shows an approximation for π accurate up to six decimal places, matches the value of $\pi^{1/3}$ shown by my calculator.

IRRATIONAL EXPONENTS

Irrational exponents add a different type of wrinkle when exploring powers. The method I described above will not help me to narrow the values in the same way. I will first explore 3^π . According to my calculator, the value is approximately 31.5442807. I know the value lies in the range of $3^{3.1416} > 3^\pi > 3^{3.1415}$. Since π is an irrational number, I also know that I can never pinpoint the exact value of 3^π , but what I can do, similar to before, is limit the range of possible values for 3^π by eliminating values greater than 3^π and eliminating values less than 3^π . I began my chart with knowing that in using 3 as an exponent, 3^3 is definitely a value less than 3^π ; likewise, 3^4 is a value greater than 3^π . My chart continues with the too low column being used as the exponent for 3 and in each row, I added one more digit of π . My 'Too High' column consists of values slightly higher than the 'Too Low' column to more clearly define the limits of where 3^π must lie. Since π is irrational, its decimal expansion never terminates, so the 'Too High' column will always be greater than π itself.

Too Low	Too High	3^Too Low	3^Too High
3	4	27	81
3.1	3.2	30.1353257	33.63473537
3.14	3.15	31.48913565	31.83698643
3.141	3.142	31.52374901	31.55840042
3.1415	3.1416	31.54106996	31.54453529
3.14159	3.1416	31.54418874	31.54453529
3.141592	3.141593	31.54425805	31.54429271
3.1415926	3.1415927	31.54427884	31.54428231
3.14159265	3.14159266	31.54428058	31.54428092
3.141592654	3.141592655	31.54428071	31.54428075

The values in the third and fourth columns of are increasing and approaching 3^π from below, and the values in the fourth column are decreasing and approach 3^π from above. Based on my chart, I've determined that $31.54428071 > 3^\pi > 31.54428075$, which is consistent with the value I determined from my calculator, which was 31.5442807.

The definition of 3^π is $3^\pi = \lim_{r_j \rightarrow \pi} 3^{r_j}$

- $r_0 = 3$
- $r_1 = 3.1$
- $r_2 = 3.14$
- $r_3 = 3.141$
- $r_4 = 3.1415$
- $r_5 = 3.14159$

..... ,note that each r_j is irrational.

EVALUATING b^π

A more general definition for b^π , where b is any positive real number, can be given using the same process as outlined above in defining 3^π . It is necessary to approximate with rational numbers and to narrow the values in between which b^π occurs.

Log₁₀ (x) and 3^π

Logarithms are useful when exponents are unknown. According to sosmath.com, exponents and logarithms share a relationship that can best be described in the following way: Since logarithms are nothing more than exponents, the rules of exponents can be used with logarithms. It is reasonable to expect the rules for exponents to be somehow reflected in the properties of logarithms. An example of a logarithm is $\log_3(b)$, which denotes the power one must raise 3 to in order to get the number b . That is, $3^{\log_3(b)} = b$, by definition of $\log_3(b)$. Logarithmic functions are the inverses of exponential functions. I will consider their use as I attempt to approximate 3^π . First, on my calculator, I find $\log_{10}(3)$ to be approximately .4771212547 Later, I will verify this using a table.

3^π can be found in the following way::

$3 = 10^a$, where, by definition, $a = \log_{10}(3) = \text{Log}(3)$ (on most calculators), which is also called the “common log” of 3. Thus, $3^\pi = 10^{a\pi}$. I can also look at this in the following way:

3^π	$10^a = 3$	$10^{a\pi} = 3^\pi$
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To estimate the value of $a = \text{Log } 3$, I will create a table. My first column contains powers, and my second column contains 10 raised to the corresponding power in the first column. When I achieve a value close to 3 in the second column, the corresponding power in the first column will be close to the common log value $\text{Log}3$.

x	10^x
0	1
0.01	1.023292992
0.02	1.047128548
0.03	1.071519305
0.04	1.096478196
0.05	1.122018454
0.06	1.148153621
0.07	1.174897555
0.08	1.202264435
0.09	1.230268771
0.1	1.258925412
0.11	1.288249552
0.12	1.318256739
0.13	1.348962883
0.14	1.380384265
0.15	1.412537545
0.16	1.445439771
0.17	1.479108388
0.18	1.513561248
0.19	1.548816619
0.2	1.584893192
0.21	1.621810097
0.22	1.659586907
0.23	1.698243652
0.24	1.737800829
0.25	1.77827941
0.26	1.819700859
0.27	1.862087137
0.28	1.905460718
0.29	1.9498446
0.3	1.995262315
0.31	2.041737945
0.32	2.089296131
0.33	2.13796209
0.34	2.187761624
0.35	2.238721139

0.36	2.290867653
0.37	2.344228815
0.38	2.398832919
0.39	2.454708916
0.4	2.511886432
0.41	2.570395783
0.42	2.630267992
0.43	2.691534804
0.44	2.754228703
0.45	2.818382931
0.46	2.884031503
0.47	2.951209227
0.48	3.01995172

Thus, $\text{Log}(3)$ is between .47 and .48. The second chart will attempt to narrow the range for $\text{Log}3$ further, beginning with .471 and, once again, looking for 3 in the second column.

<u>x</u>	<u>10^x</u>
0.47	2.951209227
0.471	2.958012467
0.472	2.96483139
0.473	2.971666032
0.474	2.978516429
0.475	2.985382619
0.476	2.992264637
0.477	2.999162519
0.478	3.006076303

Thus, $\text{Log}(3)$ is between .477 and .478. The next chart will attempt to narrow even further beginning with .477 and, once again, looking for 3 in the second column.

<u>x</u>	<u>10^x</u>
0.477	2.999162519
0.4771	2.999853181
0.4772	3.000544002

Thus, $\text{Log}(3)$ is between .4771 and .4772. My final chart will attempt to narrow $\text{Log}(3)$ to four decimal places. This chart will begin with .4771 and end when 3 is in the second column.

<u>x</u>	<u>10^x</u>
0.4771	2.999853181
0.47711	2.999922256
0.47712	2.999991333
0.47713	3.000060411

This is a numeric way to find $a = \text{Log}(3)$. Finally, to approximate 3^π , we simply multiply our approximation for $a = \text{Log}(3)$ by π , which gives us approximately 1.498948103.

Then, I will compute 10 to this power—the result is 31.54627632. This value is very close to the value I determined using integer roots and integer powers in the previous section when approximating 3^π .

The manner outlined in this section gives a way to estimate 3^π , and likewise b^c in general, using only tables of powers of 10. Imagine that you have pages of tables of 10^x , where x runs from 0 to 100 in increments of .001. With such a table, you could look up the value of $a = \text{Log}(b)$ in the table, then multiply a by c and look that exponent up in the table to find $10^{a \cdot c} = b^c$. This reduces finding b^c to working with one set of tables.

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