

**Master of Arts in Teaching (MAT)  
Masters Exam**

**Jessica Fricke**

In partial fulfillment of the requirements for the Master of Arts in Teaching with a  
Specialization in the Teaching of Middle Level Mathematics in the Department of  
Mathematics.

Jim Lewis, Advisor

July 2007

# **Extending a Finite Sequence**

**Jessica Fricke**

July 2007

One of the most common mistakes in school mathematics is to list a few terms of a sequence and ask “What term comes next?” For example, a teacher may write:

1      2      4      ...

and ask what term comes next? The best answer is “almost anything could be the next term.” But people often find that answer unsatisfying. It helps, instead, to give an example. Perhaps, one might respond by saying:

“The next number might be 8, but then again, it might be 7.”

This answer might, of course, result in an angry teacher, so you need to be prepared with the further comment:

“My choice of 8 is based on the sequence: 1, 2, 4, 8, 16, ...,  $2^n$ , , where n is a number chosen from the sequence of whole numbers, 0, 1, 2, 3, 4, ..., n, ....

But then the choice could be 7, if we are thinking of the sequence: 1, 2, 4, 7, 11, ...,  $\frac{n^2}{2} + \frac{n}{2} + 1$ , ..., where n is chosen from the whole numbers, 0, 1, 2, 3, ..., n, ....”

The point we want to make is that there can be more than one perfectly reasonable sequence that could continue a finite set of numbers. The teacher has no right to expect the student to guess the “correct” continuation.

Have this discussion with a math teacher friend and you may encounter a very frustrated friend who asks, “Aren’t we supposed to teach students to look for patterns?” Quickly reassure them that “Yes, we should.” But continue and say “But we must explain ourselves carefully so that what we say is good mathematics.”

Let’s start with an example: 3, 10, 18, 27, 37, 48, 60

One technique (perhaps it’s a “trick”) that is sometimes useful is to study the differences in successive terms:

3      10      18      27      37      48      60  
       7      8      9      10      11      12

Notice that in this specific case, we get a sequence of successive numbers. Let’s repeat our trick:

3      10      18      27      37      48      60  
       7      8      9      10      11      12  
       1      1      1      1      1

After using this trick twice, we get a sequence of constants (in fact, 1’s).

Here’s where we must make a careful mathematical statement. Let’s **assume** that our first set of numbers is part of a longer sequence, and that if we had the original terms and

continued this process of taking successive differences, then **we would continue to get the sequence of 1's in the third row.** Does this give us enough information to figure out what the next few terms of our first sequence would be?

Yes. Here's the idea: First continue the sequence of 1's and then "rebuild" the row above it by adding the 1 to the number above it, and then rebuild the row above that:

3	10	18	27	37	48	60	?	?	?	
7		8	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1	1	

And,

3	10	18	27	37	48	60	73	87	102
7		8	9	10	11	12	13	14	15
	1	1	1	1	1	1	1	1	

Using the assumptions we imposed on the sequence, the next few terms are 73, 87, 102.

Just for the record, which sequence would result from this method if we started with 1, 2, 4? If we make the assumption that we get all 1's in the third row, this method will result in selecting "7" as the next number in the sequence. Here are the details:

1	2	4	7	11	16
	1	2	3	4	5
	1	1	1	1	1

But, can we go a step further and find a mathematical formula for the sequence we are finding, i.e., for 3, 10, 18, 27, 37, 48, 60, (and) 73, 87, 102, ...?

Yes, we claim that one and only one function,  $f(n)$ , satisfies our condition of having a sequence of 1's in the third row of the "differences chart." Our candidate for an answer is:

$$f(n) = 1C(n,2) + 7C(n,1) + 3C(n,0)$$

To be precise about our notation, we start our count at "n=0" and, we use the convention that  $C(0,2) = C(1,2) = C(0,1) = 0$ . We want to check that, in fact, this function matches our sequence, but, before we do, it may be helpful to point out that the coefficients we use, namely, 1, 7, 3, amount to the first entries in our "differences chart" but in reverse order.

It might be helpful to recall some information from Math 805T. First,  $C(n,r)$  is notation for the number of ways that we can choose "r" things from a set of "n" things if the order doesn't matter. For example, if your class has 18 students and you wanted to choose 5 students for a special committee, how many different ways can you choose those 5 students?



Let's try n=7:

$$\begin{aligned}
 f(n) &= 1C(n,2) + 7C(n,1) + 3C(n,0) \\
 f(7) &= 1C(7,2) + 7C(7,1) + 3C(7,0) \\
 f(7) &= 1 * 21 + 7 * 7 + 3 * 1 \\
 f(7) &= 21 + 49 + 3 \\
 f(7) &= 73
 \end{aligned}$$

3      10      18      27      37      48      60      73

Let's try n=8:

$$\begin{aligned}
 f(n) &= 1C(n,2) + 7C(n,1) + 3C(n,0) \\
 f(8) &= 1C(8,2) + 7C(8,1) + 3C(8,0) \\
 f(8) &= 1 * 28 + 7 * 8 + 3 * 1 \\
 f(8) &= 28 + 56 + 3 \\
 f(8) &= 87
 \end{aligned}$$

3      10      18      27      **37**      48      60      73      87

Let's find out where the C(n,k) in the function comes from. Let's investigate the successive differences of C(n,0), C(n,1), C(n,2), C(n,3), C(n,k).

C(n,0) is the sequence: 1 1 1 1 1 1 1 ...

Note that this sequence is what we will call the 1<sup>st</sup> diagonal in Pascal's triangle. Since this is a constant sequence, if we take successive differences, we get all zeros.

C(n,1) is the sequence: 0 1 2 3 4 5 6... Note that this sequence is what we will call the 2<sup>nd</sup> diagonal in Pascal's triangle. Let's find the successive differences of the sequence C(n,1).

0 1 2 3 4 5 6...  
1 1 1 1 1 1

Here we see the constant sequence is made of 1's. Note that the constant sequence occurs in the *first* difference after the initial sequence.

C(n,2) is the sequence: 0 0 1 3 6 10 15... Note that this sequence is what we will call the 3<sup>rd</sup> diagonal in Pascal's triangle. Let's find the successive differences of the sequence C(n,2).

0 0 1 3 6 10 15  
0 1 2 3 4 5  
1 1 1 1 1

Here we see the constant sequence is made of 1's. Note that the constant sequence occurs in the *second* difference after the initial sequence.

$C(n,3)$  is the sequence: 0 0 0 1 4 10 20... Note that this sequence is what we will call the 4<sup>th</sup> diagonal in Pascal's triangle. Let's find the successive differences of the sequence  $C(n,3)$ .

```

0 0 0 1 4 10 20
  0 0 1 3 6 10
    0 1 2 3 4
      1 1 1 1

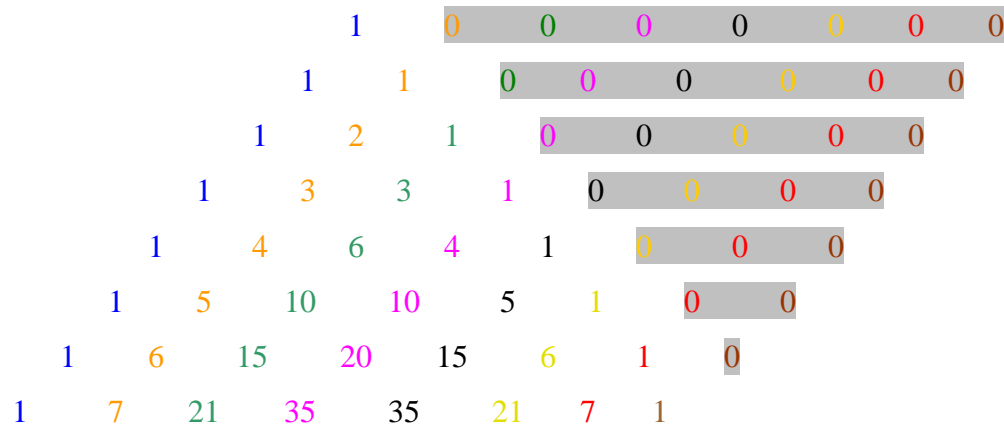
```

Here we see the successive difference is 1. Note that the constant sequence occurs in the *third* difference after the initial sequence.

In summary we can say that the number of successive differences that need to be taken in order to reach a constant sequence is equal to  $k$  in  $C(n,k)$ .

We can look at this in terms of Pascal's triangle to investigate the successive difference of any sequence  $C(n,k)$ . Note we have included zeroes from the sequences in front of some "diagonals" in Pascal's triangle below. They correlate with the terms for  $C(n,r)$  when  $n < r$ . They are shaded and indicate they are not part of Pascal's triangle and are only there for notation purposes. We have shaded them to show they are not included in the triangle. We typically look at Pascal's triangle in terms of addition. An important fact we learned in Math 805T was:

$$C(n,k) = C(n-1,k-1) + C(n-1,k).$$



C(n,7)	0	0	0	0	0	0	0	0	1						
C(n,6)		0	0	0	0	0	0	0	1	7					
C(n,5)			0	0	0	0	0	1	6	21					
C(n,4)				0	0	0	1	5	15	35					
C(n,3)					0	0	1	4	10	20	35				
C(n,2)						0	0	1	3	6	10	15	21		
C(n,1)							0	1	2	3	4	5	6	7	
C(n,0)									1	1	1	1	1	1	1

In the diagram above we are looking at Pascal's triangle as subtraction rather than addition. In other words,  $C(n,r) - C(n-1,r) = C(n-1,r-1)$  in terms of the  $r^{\text{th}}$  diagonal. That means that the successive difference in terms of the  $r^{\text{th}}$  diagonal gives the  $r-1$  diagonal. We can see that the constant sequence comes after 'k' successive differences. In other words, the successive difference of  $C(n,3)$  in pink becomes a constant sequence of 1 after the third successive difference. The successive difference of  $C(n,2)$  in green becomes a constant sequence of 1 after the second successive difference.

Let's investigate what will happen if we take the successive differences of any of these diagonals multiplied by some number. If we have  $5C(n,4)$  that means we would take the diagonal  $C(n,4)$  and multiply each of the terms in it by 5.

5C(n,4)					0	0	0	0	5	25	75	175			
5C(n,3)						0	0	0	5	20	50	100	175		
5C(n,2)							0	0	5	15	30	50	75	105	
5C(n,1)								0	5	10	15	20	25	30	35
5C(n,0)									5	5	5	5	5	5	5

Notice that to become a constant sequence, the successive difference still had to be taken k-times or 4 times after the initial  $C(n,4)$  sequence. Note that because we multiplied  $C(n,4)$  by 5, all of the differences we found are 5 times their original value. That means the constant sequence is comprised of 5's rather than 1's.

Let's consider what would happen if we reversed this process. What steps do we take to arrive at the sequence  $5C(n,4)$  from the constant sequence. First the constant sequence would consist of 5's. It would take four steps of differences to get to the constant sequence and all of the terms on the left end of each row is zero.

5C(n,0)									5	5	5	5	5	5	5					
5C(n,1)										0	5	10	15	20	25	30	35			
5C(n,2)											0	0	5	15	30	50	75	105		
5C(n,3)												0	0	0	5	20	50	100	175	
5C(n,4)													0	0	0	0	5	25	75	175

We would then expect to see this pattern continue with another example. If I investigate the successive difference of  $7C(n,2)$ , I would expect to see the constant sequence come up after the second successive difference. I would also expect to see the all of the differences including the successive to be seven times the original values.

Let's see what happens:

$7C(n,2)$	0	0	7	21	42	70	105	147
$7C(n,1)$	0	7	14	21	28	35	42	49
$7C(n,0)$		7	7	7	7	7	7	7

There it is! The constant sequence does appear in the second sequence after the initial  $C(n,2)$  sequence and the constant sequence is made up of 7's. Therefore, the successive difference of  $a^*C(n,k)$  gives  $a^*C(n,k-1)$ .

Again it works in reverse. To arrive at the sequence  $7C(n,2)$  from the constant sequence, we know that the constant sequence consists of 7's. It would take two steps of differences to get to the constant sequence and all of the terms on the left end of each row is zero.

$7C(n,0)$		7	7	7	7	7	7	7	7
$7C(n,1)$	0	7	14	21	28	35	42	49	
$7C(n,2)$	0	0	7	21	42	70	105	147	

What will happen if we take the sum of the two terms  $5C(n,4)$  and  $7C(n,2)$ . Remember the sequences were:

$5C(n,4):$	0	0	0	0	5	25	75	175
$7C(n,2):$	0	0	7	21	42	70	105	147
$5C(n,4) + 7C(n,2):$	0	0	7	21	47	95	180	322

How many successive differences will we have to take before we find a constant sequence? What will that constant sequence be made of?

$5C(n,4) + 7C(n,2)$	0	0	7	21	47	95	180	322
$5C(n,3) + 7C(n,1)$	0	7	14	26	48	85	142	
$5C(n,2) + 7C(n,0)$		7	7	12	22	37	57	
$5C(n,1)$		0	5	10	15	20		
$5C(n,0)$			5	5	5	5		

It is very important to note that at each stage of the difference chart, the " $i^{\text{th}}$ " row is the sum of the " $i^{\text{th}}$ " row of the difference chart for  $5C(n,4)$  and the " $i^{\text{th}}$ " row of the difference chart for  $7C(n,2)$ . Thus, the third row consists of the row 7, 7, 7, 7, 7, 7, 7, ... added to the row 0, 0, 5, 15, 30, 50, ... Because this is a complete decomposition of the terms, we get a 0 in the first entry of the next row before finally getting the sequence of 5's. As before, we can reverse the process by noting that we have a 7 in the outside term of the 3<sup>rd</sup> row (i.e. after two steps), a 5 in the outside term of the 5<sup>th</sup> row of constants and 0's elsewhere. This enables us to "rebuild" the original sequence as  $5C(n,4) + 7C(n,2)$ .

$$\begin{aligned}
 f(n) &= 5C(n,4) + 0C(n,3) + 7C(n,2) + 0C(n,1) + 0C(n,0) \\
 &= 5C(n,4) + 7C(n,2)
 \end{aligned}$$

We can try and see that it works:

$$f(n) = 5C(n,4) + 0C(n,3) + 7C(n,2) + 0C(n,1) + 0C(n,0)$$

$$f(3) = 5C(3,4) + 0C(3,3) + 7C(3,2) + 0C(3,1) + 0C(3,0)$$

$$f(3) = 5*0 + 0*1 + 7*3 + 0*3 + 0*1$$

$$f(3) = 0 + 0 + 21 + 0 + 0$$

$$f(3) = 21$$

0 0 7 21 47 95 180 322

Let's work this backwards and *start* with a successive difference and build a sequence from there:

			3	3	3	3	3	3		
		0	3	6	9	12	15	18		
	5	5	8	14	23	35	50	68		
	4	9	14	22	36	59	94	144	212	
1	5	14	28	50	86	145	239	383	595	← sequence

If we have successfully unraveled how this works, we expect the function that would work here will be:  $f(n) = 3C(n,4) + 0C(n,3) + 5C(n,2) + 4C(n,1) + 1C(n,0)$

Let's try it:

$$f(6) = 3C(6,4) + 0C(6,3) + 5C(6,2) + 4C(6,1) + 1C(6,0)$$

$$f(6) = 3*15 + 0*20 + 5*15 + 4*6 + 1*1$$

$$f(6) = 45 + 0 + 75 + 24 + 1$$

$$f(6) = 145$$

1 5 14 28 50 86 145 239 383 595

It works for  $f(6)$  and if one checks a few other terms they will see that it works for those terms as well.

$$f(1) = 3C(1,4) + 0C(1,3) + 5C(1,2) + 4C(1,1) + 1C(1,0)$$

$$f(1) = 3*0 + 0*0 + 5*0 + 4*1 + 1*1$$

$$f(1) = 0 + 0 + 0 + 4 + 1$$

$$f(1) = 5$$

1 5 14 28 50 86 145 239 383 595

$$f(9) = 3C(9,4) + 0C(9,3) + 5C(9,2) + 4C(9,1) + 1C(9,0)$$

$$f(9) = 3*126 + 0*84 + 5*36 + 4*9 + 1*1$$

$$f(9) = 378 + 0 + 180 + 36 + 1$$

$$f(9) = 595$$

1 5 14 28 50 86 145 239 383 595

Thus, the good news is that it seems we have "broken the code."

Why does it work?

It is important to remember that we built in the assumption that our method of successive differences gets us to a constant sequence. This rules out lots of sequences for which the successive differences would never get us to a constant sequence.

I have learned that if you are able to take successive differences of a sequence and arrive at a constant sequence then you are able to build a function that will allow you to find any term 'n' in that sequence. You can build that function using what we have learned about Pascal's triangle. If you are able to arrive at a constant sequence then the initial sequence is related to Pascal's triangle. Pascal's triangle is made of successive differences and consists of "diagonals" that illustrate those successive differences. Each "diagonal" is a difference of the "diagonal" before. As you take differences of each "diagonal" you will eventually get down to the first "diagonal" which is a constant sequence that is made of 1's.

Each "diagonal" in Pascal's triangle is a sequence  $C(n,r)$ . That means we are taking successive differences of combination numbers. When we multiply a sequence  $C(n,r)$  by any whole number greater than 1 ('a'), then we are multiplying all of the terms in the "r + 1 diagonal" by 'a'. When we take successive differences of that sequence all of the terms are multiplied by 'a' as well. We will arrive at a constant sequence that is made up of 1's multiplied by the number 'a'.

$aC(n,r)$  is the general sequence:

$$\begin{array}{cccccc} aC(n,r) & aC(n+1,r) & aC(n+2,r) & aC(n+3,r) & aC(n+4,r) & \dots \\ aC(n-1,r-1) & aC(n+1,r-1) & aC(n+2,r-1) & aC(n+3,r-1) & \dots & \end{array}$$

The number of successive differences we must take to get to the constant sequence will determine the 'r' value in  $C(n,r)$  that we start with in our function. The leftmost terms in the successive difference chart will determine the coefficients in our function because they are what we are to multiply that particular "diagonal" by. When we find the sum of all of those "diagonals" we will arrive at the initial sequence.