The Polygon Game

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In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.

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December 2007

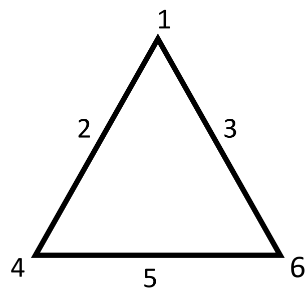
The Polygon Game - Take a regular, *n*-sided polygon (i.e. a regular *n*-gon) and the set of numbers, {1, 2, 3, …, (2*n*-2), (2*n*-1), 2*n*}. Place a dot at each vertex of the polygon and at the midpoint of each side of the polygon. Take the numbers and place one number beside each dot. A *side sum* is the sum of the number assigned to any midpoint plus the numbers assigned to the vertex on either side of the midpoint. A *solution* to the game is any polygon with numbers assigned to each dot for which all side sums are equal, i.e. for which you have *equal side sums*. The most general problem we might state is, "Find all solutions to The Polygon Game."

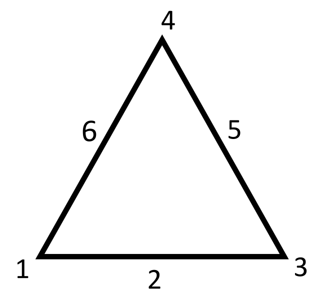
For this paper I will use The Polygon Game to create an extended activity for a middle school mathematics class. As I work through this paper I will explore solutions to the triangle game, the square game, the pentagon game, the hexagon game and the n-gon game. From these explorations I will discuss different topics such as equality, duality and possible side sum solutions.

The Triangle Game

To begin my discussion about The Polygon Game, I will first look at the triangle game because a triangle is the simplest polygon. I know that I have three vertices and three midpoints that need to have a number assigned to them. Therefore I will be using the numbers 1, 2, 3, 4, 5, and 6 to compute side sums of the triangle game.

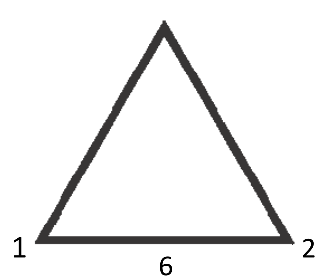
In order to find all of the possible side sums of the triangle game I must consider what the smallest and largest side sums could be. This will give me a range of possible side sums from which I can check for solutions. For example, the sum of the smallest three numbers is 6 because 1 + 2 + 3 = 6. This means that it is not possible to have a side sum smaller than 6 because I need to use 3 of my numbers on each side of the triangle. Does this mean that 6 is the smallest side sum for which there is a solution? Additionally, the largest three numbers are 6, 5 and 4 so 6 + 5 + 4 = 15. Does this mean that 15 is therefore the largest side sum for which there is a solution? Further investigations lead me to the conclusion that there is no solution with 6 or 15 as the common side sum, because in finding a solution to the triangle game, all of the side sums must be equal. It is not possible to put all three of either the smallest or the largest numbers on the same side of the triangle and have all three sides remain equal. Here are examples:



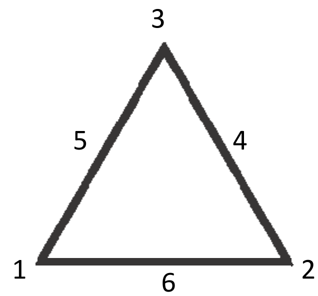


Now that we have ruled out 6 and 15 as the smallest and largest side sums for which there are solutions, how can I narrow this interval? First consider the smallest number, 1. We know that the number 1 must appear somewhere on the triangle. If we place it with the numbers 5 and 6, the two largest numbers that will be placed on the triangle, then since 1 + 5 + 6 = 12, we know 12 is the largest possible side sum that needs to be checked for solutions. Similarly, if we now look at the highest number, 6, and place this with the numbers 1 and 2, the two smallest numbers, then since 1 + 2 + 6 = 9, we know 9 is the smallest possible side sum that needs to be checked for solutions. Since I now know the smallest and largest possibilities for side sums I can determine that the range of side sums that I need to check for solutions is 9 – 12.

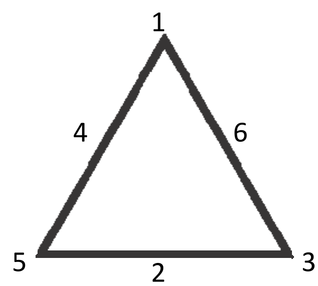
I will start with the side sum of 9 and try to find a solution. This means that I need all three sides of the triangle to sum to 9 using the numbers 1 – 6. I can randomly try numbers until I find a solution, but if there is not a solution, how can I be sure that I have tried all possibilities? This means that I need a strategy. I will start by listing all of the ways that I can make 9 using the largest number, 6. The only option that I have is 1 + 2 + 6 = 9. I can arrange these three numbers in two different ways according to the position of the 6. Either the 6 must be at the midpoint or the 6 must be at the vertex of a triangle side. However, since there is only one way to get a sum of 9 using the number 6, the 6 must be placed at the midpoint of a triangle side. So I will start with a triangle that looks like this:



Then I will consider the numbers that I have left to work with: 3, 4, and 5. I need to arrange these three numbers in a way that the other two sides of the triangle also add up to 9. I will begin on the side that has a 1 at the vertex. How can I get a total of 9 using 1 and two of the above numbers? Since 3 + 5 + 1 = 9 I will place the 3 at the midpoint and the 5 at the last vertex. Then I check the third side and since 5 + 4 + 2 = 10, I know this will not work. Now I must start rearranging the numbers to see if I can find a solution. I know that I have to leave the 1, 6 and 2 where they are, so I will look again at the side with the 1, 3 and 5. I will switch the 3 to the vertex and the 5 to the midpoint to obtain 3 + 5 + 1 = 9. Now checking the last side we have 3 + 4 + 2 = 9. I have found a solution! This is what it looks like:



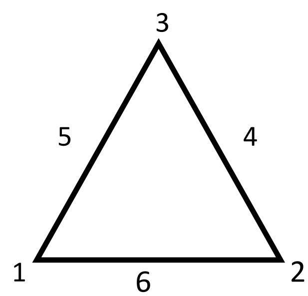
Now that I have found a solution for the side sum 9, I will move on and try to find a solution for the side sum 10. Following a similar strategy to the one described above, I arrived at the following solution for a side sum of 10:



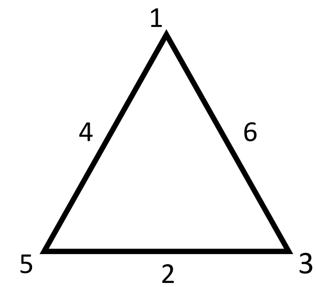
This brings me to my general strategy: First I listed all the possible ways that I could use my numbers (ex: 1 – 6) to equal my side sum (ex: 9, 10, 11 or 12) by starting with the highest number (ex: 6). Second I determined the location for this high number, which could be placed at the midpoint of a side if there was only one way to make the sum or at the vertex if there were two ways to make the sum. Third I chose a side to work from and used the remaining numbers to come up with a combination that equaled my side sum. If there was more than one option I tried them all to make sure that I found or eliminated all possible solutions for every side sum.

At this point I could continue implementing my same strategy to try to find solutions for the side sums of 11 and 12, but I could also use the concept of duality. Sally and Sally (2003) describe duality in this way: If I add up all of the numbers that can be placed around the triangle (1 – 6) I will get a sum of 21. I know that there are solutions to the triangles with side sums of 9 and 10. I will then use these triangles to find a dual triangle. If I subtract each number around the triangle from 7, because 7 is one digit higher than 6, which is the largest number used in the triangle game, then I will be left with the dual triangle. Here are my original triangles with side sums of 9 and 10 and their duals:

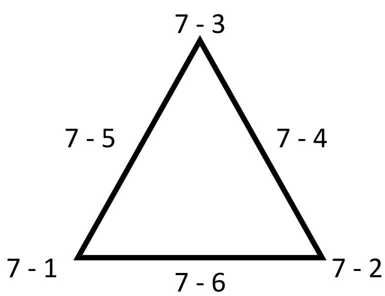
Side Sum 9



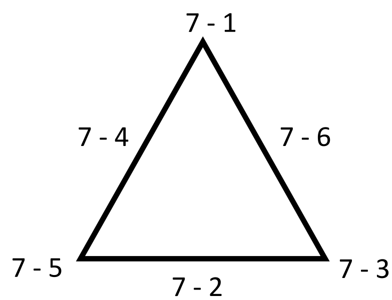
Side Sum 10



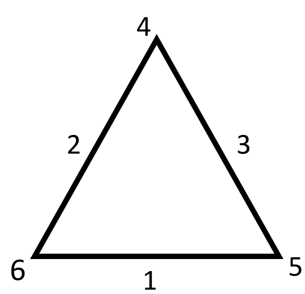
Applying the concept of duality



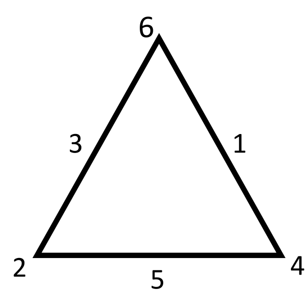
Applying the concept of duality



Results in Side Sum 12

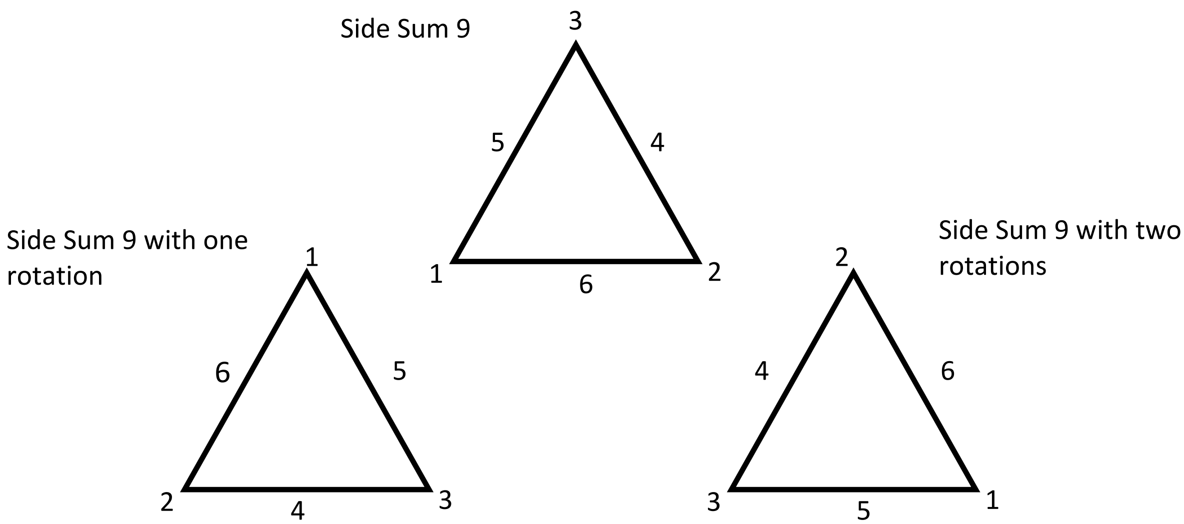


Results in Side Sum 11

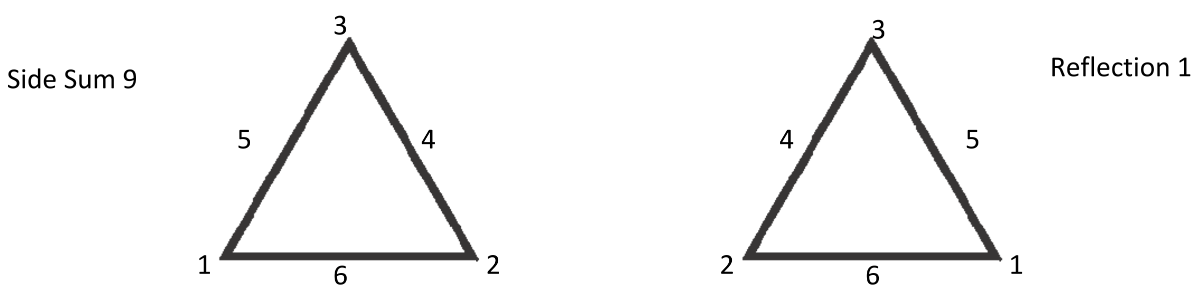


So, because 21 is the sum of all of the triangle numbers (1 – 6) and 21 – 9 = 12, the dual of the triangle with side sum 9 is the triangle with side sum 12 and vice versa. This also works for side sum 10 because 21 – 10 = 11. So the dual of the triangle with side sum 10 is the triangle with side sum 11 and vice versa. This process gives a solution because if S is the side sum for the first triangle, then each side of the new triangle adds to 7*3 – S. With this information I no longer have to search for solutions for all the possible side sums within my range. I now only have to search for solutions to the first half of the side sums within my range. Then, by the concept of duality, I can find solutions for the rest of the side sums.

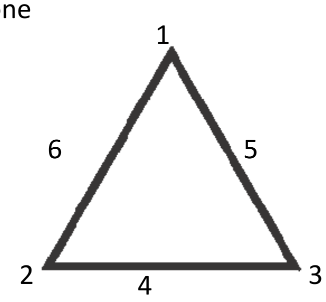
Now that I have found a solution for each sum (9 - 12) in my range of possible side sums, I have to consider the fact that there may be other solutions as well. Although I did use an organized strategy to come up with the solutions that I have found, at this point I am still not confident that I have found all of the possible solutions to the triangle game. For example, what would happen if I rotated a triangle? This is what my original solution for side sum 9 looks like along with two clockwise rotations:

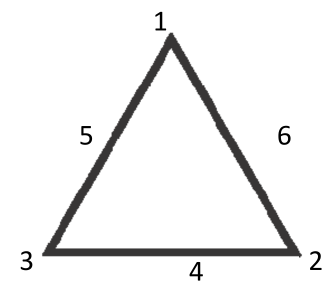


Since I have found two additional solutions to the polygon game for side sum 9 by rotating the original solution clockwise two different times, I can assume that I can also rotate the solutions to side sums 10, 11 and 12. This would produce a total of 12 solutions to the triangle game. However, I still wonder if I have found all of the solutions. For example, what would happen if I reflected a triangle? This is what my original solution for side sum 9 looks like along with a reflection from each of the two clockwise rotations:



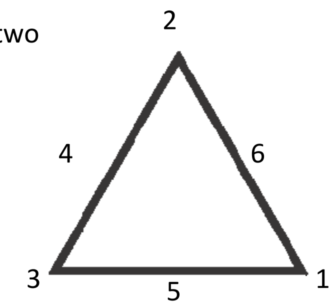
Side Sum 9 with one rotation

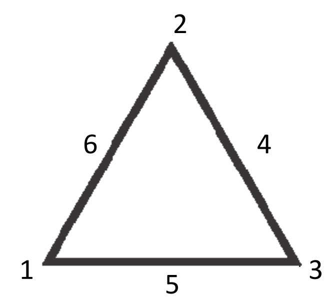




Reflection 2

Side Sum 9 with two rotations





Reflection 3

Since I have found three additional solutions to the polygon game for side sum 9 by reflecting the original solution and the two rotation solutions, I can assume that there are also three additional solutions for each of the side sums of 10, 11 and 12. This would produce a total of 12 more solutions to the triangle game for a total of 24 solutions. I am now confident that I have found all of the possible solutions to the triangle game for the following reasons: First, as I explained previously, to get a side sum of 9 I have to use 1 + 6 + 2. This leaves only one possible arrangement for the 3, 4 and 5. Any other side sum of 9 is either a rotation or a reflection of the original triangle. Second, to get a side sum of 10 I have to use 1 + 6 + 3. Similarly, this leaves only one possible choice for the 3, 4 and 5. Any other side sum of 10 is either a rotation or a reflection of the original triangle. Finally, by duality, this holds true for side sums 11 and 12.

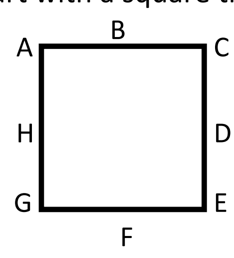
Now we ask the question, how many *distinct* solutions are there to the triangle game? In other words, are rotations and reflections really solutions to the triangle game or are they simply different variations of the original solution to each of the side sums? This depends on the definition of equality for two solutions. Definition 1: Two solutions are equal if there is a combination of rotations and reflections of the triangle so that the two solutions match exactly. Definition 2: Name each vertex and midpoint (A, B, C, D, E, F). A solution is a permutation of the numbers 1 – 6 chosen so that the sums are equal.

I will first discuss rotations. The original solution that I found to the triangle game for side sum 9 beginning with the top vertex and reading clockwise is: 3, 4, 2, 6, 1, 5. By rotating this triangle clockwise two more times I get the following solutions: 1, 5, 3, 4, 2, 6 and 2, 6, 1, 5, 3, 4. But are these two solutions really distinct from the original solution and from each other? Notice that all three solutions are similar. The only difference in them is where the initial number (for example 6) is placed on the blank triangle. The same case can be made for reflections. Once the first number is placed (for example 6 can be placed at any midpoint), then the 1 and the 2 have to be placed at the adjacent vertices on either side of the 6. Then to get the reflection all I have to do is switch the 1 and the 2. This same reflection can be done for each of the three sides of the triangle.

Thus, the number of solutions is really determined by the definition of equality. I have presented two different definitions for equality, the first resulting in 4 solutions and the second resulting in 24 solutions.

<u>The Square Game</u>

Since I have determined a complete solution to the triangle game I will move on to the square game. I need to determine what in fact are the smallest and largest possible side sums for squares? There are four vertices and four midpoints on a square so I will use the numbers 1 – 8 to make my side sums. First consider the smallest number, 1. The numbers 7 and 8 are the largest numbers that can be put with 1 and 1 + 7 + 8 = 16. Therefore 16 is the largest possible side sum that needs to be checked for solutions. Now look at the highest number, 8. The numbers 1 and 2 are the smallest numbers that can be put with 8 and 1 + 2 + 8 = 11. Therefore 11 is the smallest possible side sum that needs to be checked for solutions. Since I now know the smallest and largest possibilities for side sums I can determine that the range of side sums that I need to check for solutions is 11 – 16. However, because of duality as described in the triangle game, I only have to check half of the solutions. Then I can find the other half of the solutions by using the concept of duality. So I either have to check side sums for 11, 12 and 13 or for 14, 15, and 16. Although my possibilities seem to be cut down quite a bit, as my polygons get more complex I get more numbers and my possibilities increase dramatically. So instead of using the method of exhaustion that I used to find all of the solutions to the triangle game I will look at the problem another way. I will start with a square that looks like this:



Note that for a solution to the square game we know that 4 times the common side sum equals the sum of the numbers 1 through 8 plus the four numbers placed on the vertices (since they are shared by two sides). This can be written as

$$4S = (A + B + C + D + E + F + G + H) + (A + C + E + G)$$

So I will try my smallest possible solution, side sum 11 as determined above, in my formula.

$$4 * 11 = (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) + (A + C + E + G)$$

$$44 = 36 + (A + C + E + G)$$

$$8 = (A + C + E + G)$$

I cannot add any four of the digits 1 – 8 together to get 8. Therefore 11 is not a possible side sum for which there is a solution and by duality, 16 is not a possible side sum either.

So I will try the next possible solution, side sum 12, in my formula.

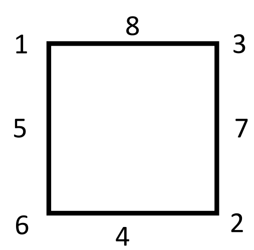
$$4 * 12 = (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) + (A + C + E + G)$$

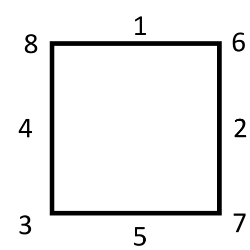
$$48 = 36 + (A + C + E + G)$$

$$12 = (A + C + E + G)$$

There are two sets of four numbers from the digits 1 – 8 that sum to 12: 1 + 2 + 3 + 6 = 12 and 1 + 2 + 4 + 5 = 12. Now I can check these two sets of numbers as vertices to see if they result in a solution with side sum 12. I know that the numbers 1 and 2 cannot be adjacent vertices because 1 + 2 + 8 = 11, which is too small. So 3 and 6 will go on the other two vertices and I have the numbers 4, 5, 7 and 8 left to place. I am able to place the numbers and find a solution. By doing so I also have a solution for the dual, which is the square with side sum 15:

Solution for Side Sum 12

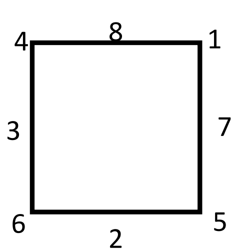


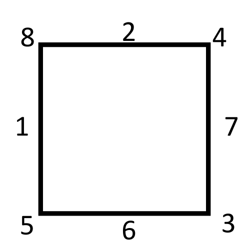


Solution for Side Sum 15

When I try the other possible solution for 12 in the same way I am unable to come up with all side sums equaling 12. Therefore there is only one solution to side sum 12. By continuing through this process I find the following two solutions for side sum 13 and use the concept of duality to find the solutions for 14:

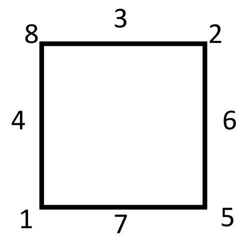
Solution one for Side Sum 13

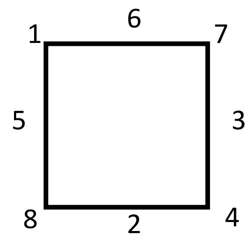




Solution one for Side Sum 14

Solution two for Side Sum 13





Solution two for Side Sum 14

Now that I have found all of the possible solutions (there are six) for the side sums 11 – 16, I could start counting rotations and reflections. Each square can be rotated three additional times clockwise and can be reflected from each vertex. This would result in a total of 48 solutions to the square game. Again, whether or not there are 6 solutions or 48 solutions depends on which definition of equality I choose.

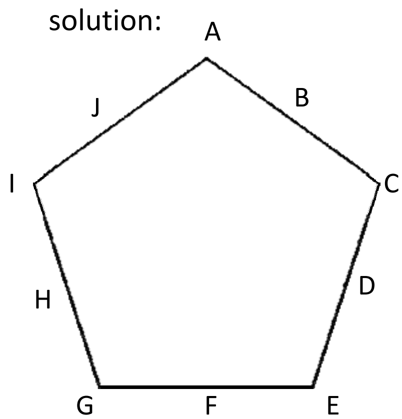
<u>The Pentagon Game</u>

Since I have determined a complete solution to both the triangle game and the square game I will move on to the pentagon game. Once again I need to determine the smallest and largest possible side sums for pentagons. There are five vertices and five midpoints on a pentagon so I will use the numbers 1 – 10 to make my side sums. I will use the same process for pentagons that I used above to find all of the solutions to the square game. Therefore 1 + 2 + 10 = 13 is the smallest side sum for which I need to check for possible solutions and 10 + 9 + 1 = 20 is the largest side sum for which I need to check for possible solutions. Once again, by duality, I only have to check half of the possible side sums for solutions, so I will check 13, 14, 15, and 16, given this pentagon and equation.

$$5S = (A + B + C + D + E + F + G + H + I + J) + (A + C + E + G + I)$$

$$5S = 55 + (A + C + E + G + I) \qquad \text{where } S = \text{side sum}$$

This leads to the following solutions listed from A – J. Note that not all possible side sums have a solution:



Side Sum 13: None

(Dual) Side Sum 20: None

Side Sum 14: 1, 10, 3, 6, 5, 7, 2, 8, 4, 9

(Dual) Side Sum 19: 10, 1, 8, 5, 6, 4, 9, 3, 7, 2

Side Sum 15: None

(Dual) Side Sum 18: None

Side Sum 16: 1, 10, 5, 2, 9, 4, 3, 6, 7, 8

And 10, 2, 4, 9, 3, 6, 7, 8, 1, 5

(Dual) Side Sum 17: 10, 1, 6, 9, 2, 7, 8, 5, 4, 3

And 1, 9, 7, 2, 8, 5, 4, 3, 10, 6

Now that I have found all of the possible solutions(there are six) for the side sums 13 – 20, I could continue by counting rotations and reflections. Each pentagon can be rotated four additional times clockwise and can be reflected from each vertex. This would result in a total of 60 solutions to the pentagon game. Again, whether or not there are 6 solutions or 60 solutions depends on which definition of equality I choose.

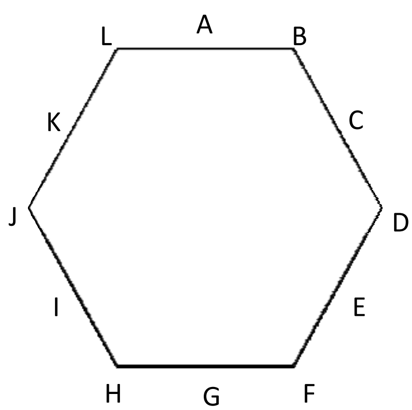
The Hexagon Game

The hexagon game will be the last polygon game that I will discuss in detail before moving on to the n-gon game. I will discuss this game so that I can establish a pattern with polygons in order to explore the n-gon game. I will solve this game just like the square game and the pentagon game. So I need to determine the smallest and largest possible side sums for hexagons. There are six vertices and six midpoints on a hexagon so I will use the numbers 1 – 12 to make my side sums. Therefore 1 + 2 + 12 = 15 is the smallest side sum for which I need to check for possible solutions and 12 + 11 + 1 = 24 is the largest side sum for which I need to check for possible solutions. Once again, by duality, I only have to check half of the possible side sums for solutions, so I will check 15, 16, 17, 18 and 19, given this hexagon and equation.

$$6S = (A + B + C + D + E + F + G + H + I + J + K + L) + (B + D + F + H + J + L)$$

$$6S = 78 + (B + D + F + H + J + L) \qquad \text{where } S = \text{side sum}$$

I came up with the following solutions listed from A – L . Note that not all possible side sums have a solution:



Side Sum 15 and 16: None

(Dual) Side Sum 24 and 23: None

Side Sum 17: 12, 3, 8, 6, 4, 7, 9, 1, 11, 5, 10, 2

(Dual) Side Sum 22: 1, 10, 5, 7, 9, 6, 4, 12, 2, 8, 3, 11

Side Sum 18: None

(Dual) Side Sum 21: None

Side Sum 19: 12, 6, 2, 11, 5, 3, 9, 7, 4, 8, 10, 1

And 12, 4, 10, 5, 8, 6, 2, 11, 1, 7, 9, 3

And 2, 5, 11, 3, 9, 7, 4, 8, 10, 1, 6, 12

And 4, 3, 9, 7, 11, 1, 10, 8, 6, 5, 2, 12

(Dual) Side Sum 20: 1, 7, 11, 2, 8, 10, 4, 6, 9, 5, 3, 12

And 1, 9, 3, 8, 5, 7, 11, 2, 12, 6, 4, 10

And 11, 8, 2, 10, 4, 6, 9, 5, 3, 12, 7, 1

And 9, 10, 4, 6, 2, 12, 3, 5, 7, 8, 11, 1

Now that I have found all of the possible solutions (there are ten) for the side sums 15 – 24, I could continue by counting rotations and reflections. Each hexagon can be rotated five additional times clockwise and can be reflected from each vertex. This would result in a total of 120 solutions to the hexagon game. Again, whether or not there are 10 solutions or 120 solutions depends on which definition of equality I choose.

<u>The n-gon Game</u>

I now have complete solutions to the triangle game, the square game, the pentagon game and the hexagon game which will help me to establish patterns in order to explore the n-gon game. In discussing the n-gon game I will introduce one pattern that runs through the polygon games and two methods for finding solutions to larger polygons.

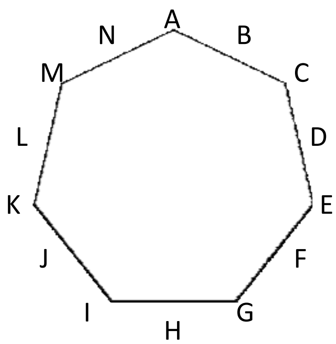
Patterns with Polygon Side Sums (minimums and maximums)

Beginning with the square game I used an algebraic approach to eliminate some values as possible side sums for which there are solutions and to predict the minimum and maximum possible values for these side sums. Then, I would experiment until I found a solution for the minimum side sum and use the concept of duality to find a solution for the maximum side sum. In the basic equation that I used the product of the side sum and the number of sides equals the sum of all of the numbers that are placed at the midpoints and vertices plus the sum of the numbers placed at the vertices (since these are shared by two sides). Letting k be the number of sides of the polygon and $v_1 – v_k$ and $m_1 – m_k$ be the numbers placed at the vertices and midpoints, respectively, we can write this equation as

$$kS = (v_1 + m_1 + v_2 + m_2 + v_3 + m_3 + \ldots + v_k + m_k) + (v_1 + v_2 + v_3 + \ldots + v_k), \text{ where } S = \text{side sum.}$$

I can continue to use this equation for increasingly larger polygons by simply plugging in the necessary information. This equation helps me to narrow down the possible side sums and combinations that I have to try in order to find solutions. I will now use the equation to find the minimum and maximum solutions to the heptagon game and the octagon game.

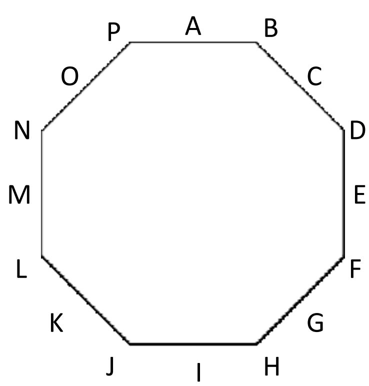
Heptagons have 7 vertices and 7 midpoints so I will use the numbers 1 – 14 to find side sums. Since 1 + 2 + 14 = 17, this is the smallest possible side sum with solutions and 14 + 13 + 1 = 28 is the largest possible side sum with solutions. Now I can use my equation to determine the lowest side sum that has a solution. Consider the standard heptagon and corresponding equation:



$$7S = (A + B + C + D + E + F + G + H + I + J + K + L + M + N)$$
$$+ (A + C + E + G + I + K + M)$$
$$7S = 105 + (A + C + E + G + I + K + M) \quad \text{where } S = \text{side sum}$$
$$7 * 17 = 105 + (A + C + E + G + I + K + M)$$
$$14 = (A + C + E + G + I + K + M)$$

Note that 17 is not a possible side sum because we cannot select seven numbers from 1 – 14 that will have a sum of 14. Similarly, 18 is also not a possible side sum, but 19 is. The solution is (listed in order from A – N): 14, 1, 13, 5, 12, 2, 11, 6, 10, 3, 9, 7, 8, 4. Therefore 19 is the smallest possible side sum solution and, by duality, 26 is the largest possible side sum.

Octagons have 8 vertices and 8 midpoints so I will use the numbers 1 – 16 to find possible side sums. Since 1 + 2 + 16 = 19, this is the smallest possible side sum for which I will check for solutions and 16 + 15 + 1 = 32 is the largest. I will now use my equation to determine the lowest side sum that has a solution. Consider the standard octagon and corresponding equation:



$$8S = (A + B + C + D + E + F + G + H + I + J + K + L + M + N + O + P)$$
$$+ (B + D + F + H + J + L + N + P)$$
$$8S = 136 + (A + C + E + G + I + K + M + O) \quad \text{where } S = \text{side sum}$$
$$8 * 19 = 136 + (B + D + F + H + J + L + N + P)$$
$$16 = (B + D + F + H + J + L + N + P)$$

Note that 19 is not a possible side sum that has a solution because we cannot select eight numbers from 1 – 16 that will have a sum of 16. By the same method 20 and 21 are also not side sums with a solution. However 22 is a possible side sum for which there are solutions because it is possible to satisfy the equation 8*22 = 136 + (B + D + F + H + J + L + N + P). In fact, I have found a solution for 22 but chose not list it in this paper. Therefore 22 is the smallest possible side sum for which there is a solution and, by duality, 29 is the largest possible side sum.

With the help of the equation above I was able to find the minimum and maximum solutions to the triangle game, the square game, the pentagon game, the hexagon game, the heptagon game and the octagon game. Consider the following chart:

Polygon	**Minimum Side Sum**	**To find the next Minimum**	**Maximum Side Sum**	**To find the next Maximum**
Triangle	9	+3	12	+3
Square	12	+2	15	+4
Pentagon	14	+3	19	+3
Hexagon	17	+2	22	+4
Heptagon	19	+3	26	+3
Octagon	22		29	

The chart should be read: minimum side sum of 9 + 3 (to find the next minimum) results in 12.

Conjecture:

Notice the patterns that result in the chart above. To get the next minimum side sum for which there is a solution, continue the +3, +2, +3 pattern. Similarly, to get the next maximum side sum for which there is a solution, continue the +3, +4, +3 pattern. This pattern provides a way that is different from the equation above to find the minimum and maximum side sum solutions.

Solutions to Polygons with an Odd Number of Sides-

I am going to present a solution for a pentagon, a heptagon and a nonagon. Then from these solutions I will generalize my solutions so that I will be able to find a side sum solution for any odd sided polygon.

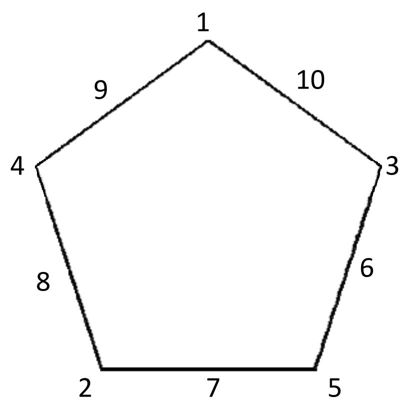
The solutions will make use of the following notation:

k = the number of vertices (must be odd)

n = (k – 1)/2; since k is odd it can be written as k = 2*n + 1 where n is an integer

Capital letters correspond with the numbers placed at the vertices of the polygon; the numbers situated between the vertices in the table are those placed at the midpoints.

Pentagon Solution:

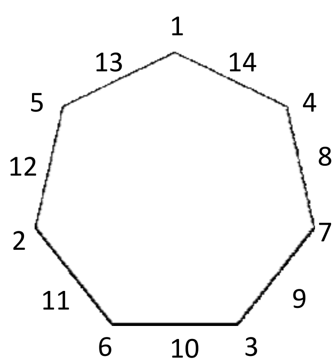


k = 5 n = 2 General Formula: 5 = 2*2 + 1

A	B	C	D	E
1	3	5	2	4
10	6	7	8	9

To determine the numbers placed at the vertices, we begin with vertex A, or the number 1, and add n each time and reduce in modulo k as needed. For example 1 + 2 = 3, 3 + 2 = 5, 5 + 2 = 7 ≡ 2 (mod 5), and 2 + 2 = 4. A result is that this side sum uses the smallest numbers on the vertices and it is therefore the smallest possible side sum. The midpoints are 2k, k + 1, k + 2, k + 3, and 2k – 1. Now I will consider the side sum 1 + 10 + 3 = 14. When I rewrite this side sum in terms of n I get 1 + 2*(2*n + 1) + (n + 1) = 5*n + 4. The side sums for this pentagon are 5*2 + 4 = 14. Another way to look at this is to substitute n = (k – 1)/2 for n in the expression 5*n + 4. This will be (5*k + 3)/2 and will equal (5*5 + 3)/2 = 14.

Heptagon Solution:



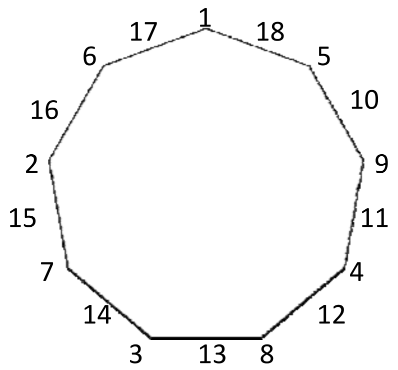
k = 7 n = 3 General Formula: 7 = 2*3 + 1

A	B	C	D	E	F	G
1	4	7	3	6	2	5
14	8	9	10	11	12	13

Once again, beginning with vertex A, or the number 1, I add n to find each subsequent vertex and reduce modulo k. For example 1 + 3 = 4, 4 + 3 = 7, 7 + 3 = 10 ≡ 3 (mod 7), etc. Once again, the result is that this side sum uses the smallest numbers on the vertices and it is therefore the smallest possible side sum. The midpoints are 2k, k + 1, k + 2, k + 3, … 2k – 1.

Now I will consider the side sum 1 + 14 + 4 = 19. When I rewrite this side sum in terms of n I still get 1 + 2*(2*n + 1) + (n + 1) = 5*n + 4. The side sums for this heptagon are 5*3 + 4 = 19 or (5*k + 3)/2 and will equal (5*7 + 3)/2 = 14.

<u>Nonagon Solution:</u>



k = 9 n = 4 General Formula: 9 = 2*4 + 1

A	B	C	D	E	F	G	H	I
1	5	9	4	8	3	7	2	6
18	10	11	12	13	14	15	16	17

In this case the side sums are 24 = 5*4 + 4 or 24 = (5*9 + 3)/2.

<u>k-gon:</u>

Now that I have shown three examples, I will make a generalization. I know that the general formula is k = 2*n + 1 and that side sums should be 5*n + 4 or (5k + 3)/2. Once again, consider the letters A – Z to be vertices.

A	B	C	D	E	…	Y	Z
1	1 + n	1 + 2n	1 + 3n		…	1 + (k-2)n	1 + (k-1)n
2k	k + 1	k + 2	k + 3		…	2k-2	2k-1

In the general solution above, each vertex number is reduced modulo k (except that we use k rather than 0 for one vertex). This assures me that the vertices use the smallest possible values, i.e., 1, 2, 3, … , k. However, I need to be certain that each vertex is different and that as a result I get all the numbers from 1 to k on the vertices.

Suppose there exists integers i and j where $0 \leq i,j \leq k-1$ such that $1 + i*n \equiv 1 + j*n \pmod{k}$, or equivalently $i*n \equiv j*n \pmod{k}$. Then $2*i*n \equiv 2*j*n \pmod{k}$ which, by substituting n = (k – 1)/2, implies $i(k-1) \equiv j(k-1) \pmod{k}$. Since $k \equiv 0 \pmod{k}$, this implies $i \equiv j \pmod{k}$. Both i and j are less than k therefore they must be equal. Therefore, the general formula for finding vertices gives the smallest numbers on the vertices, which provides the smallest possible side sum solution for a polygon.

Now I will consider the side beginning with vertex A. 1 + 2k + (1 + n) = 1 + 2*(2*n + 1) + (n + 1) = 5*n + 4. This is what I hoped to get. Now I will consider the second side sum beginning with vertex B, which is in both the first side sum and the second, so I will look at vertex C. The vertices A and C trade a 1 for 1 + 2n (so the side sum increases by 2n = k – 1). The midpoints also trade 2k for k + 1 (which is the same as subtracting k – 1). Therefore the net change is zero, and the side sum is the same. If I continue doing this I will trade 1 + n for 1 + 3n (so the side sum increases by 2n = k – 1). The midpoints also trade k + 1 for k + 2 (which is an increase of 1). Therefore the net change is k, but k (mod k) is the same as zero. This pattern continues for each side sum.

Therefore, for a k-gon (which has an odd number of sides) the general formula is k = 2*n + 1 and the side sum is 5*n + 4, or combining the equation and expression results in (5k + 3)/2.

The Search for Solutions to All Polygons-

Conjecture:

One solution to the polygon game for any n-gon will have a side of n + 2n + 1, where n is the number of vertices on the polygon, giving a side sum of 3n + 1. Consider the following examples, all of which are solutions to side sums in the lower half of the range of possible side sums:

Triangle: 3, 6, 1, 4, 5, 2 — Side Sum = 10

Square: 4, 8, 1, 7, 5, 2, 6, 3 — Side Sum = 13

Pentagon: 5, 10, 1, 8, 7, 6, 3, 4, 9, 2 — Side Sum = 16

Hexagon: 6, 12, 1, 10, 8, 4, 7, 9, 3, 5, 11, 2 — Side Sum = 19

Notice that in each example the underlined numbers represent a side sum that is consistent with the expression n + 2n + 1. I also checked to see if this would be true for an octagon and a decagon. So, I fixed the expression as a given side and checked for solutions.

Octagon: 8, 16, 1, 13, 11, 12, 2, 9, 14, 5, 6, 4, 15, 3, 7, 10 — Side Sum = 25

Decagon: 10, 20, 1, 14, 16, 11, 4, 15, 12, 13, 6, 7, 18, 8, 5, 17, 9, 3, 19, 2 — Side Sum = 31

My conjecture that a solution to the polygon game can be formed using side sum n + 2n + 1 works for these examples. While this suggests that the conjecture may hold for many more polygons, proving that it holds true in every case is beyond the scope of this paper.

With this last conjecture, my exploration of the polygon game comes to an end. In summary, I have determined the solutions to the triangle game, the square game, the pentagon game and the hexagon game. I then used that information to find patterns that allowed me to explore n-gons in two different ways, from which two solutions to any odd sided polygon were determined along with one solution to any even sided polygon. In addition I applied the concept of duality to double the number of solutions obtained for any polygon.

References

Sally, Judith D., Sally, Paul J., *TriMathlon: A Workout Beyond the School Curriculum*. 2003. A. K. Peters Ltd. Natick, MA.