

**Master of Arts in Teaching (MAT)
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Introduction

The number zero is a very powerful tool in mathematics that has many different applications and rules. An interesting fact about the number zero is that according to our calendar (the Gregorian calendar), there is no “year zero” in our history. There is also no “zeroth” century as time is recorded from centuries B.C. to the 1st century A.D. However, certain calendars do have a year zero. In the astronomical year numbering system year zero is defined as year 1 BC. Buddhist and Hindu lunar calendars also have a year zero. In this paper I am going to discuss many different uses of the number zero in mathematics and in the world. I am also going to discuss the origins of the number zero.

Zero as a Numeral

I learned from Wikipedia, the word zero comes from the Arabic translation of the Sanskrit *sunya*, around 400 B.C. in India. Sunya means void or empty. The Romans translated the word between the 5th and 1st century B.C. using Latin to become *zephyrus* which meant “west wind”. The word zephyr is used today to mean a light breeze. Fibonacci, a famous Italian mathematician, who lived from 1170 to 1250, introduced this concept to Europe and the word became *zefiro* in Italian, and was then changed to *zero* in the Venetian dialect. The actual symbol 0 that we use was developed in India in the 9th century A.D.

The number zero is an important number because it serves as a place holder to help distinguish the difference between numbers like 230 and 23. The ancient Babylonians, who used a base 60 system, did not have zero as a placeholder, but instead used a separator mark to separate places. The Babylonians had 59 different numbers that fit into a single position. Each number consisted of a different combination of symbols. The first position from the right is the 60^0 or units place, the next place to the left is the 60^1 place, and so on through 60^n . The number 8 32 would be $8 \times 60^1 + 32 \times 60^0$ which would be $480 + 32$ which equals 512. The base 60 system is still used in our society today through the way that we measure time (sixty seconds in a minute, sixty minutes in a hour) and the way that we measure the degrees in a circle.

Numbers with Many Zeros

A googol is defined as 10^{100} , which can be written as a 1 followed by 100 zeros. It can also be obtained by taking 1 million to the 16th power and then multiplying that result by ten thousand. On Wikipedia, I found the term *googol* was coined in 1920 by a nine year old named Milton Sirotta, the nephew of a famous mathematician named Edward Kasner. A googol has only two prime factors, they are 2 and 5 and there are 100 of each of them.

There are many names for very large numbers that I found while looking through at the online encyclopedia called Wikipedia. I learned that a quadrillion is 10^{15} , an octillion is 10^{27} , a googolplex is $10^{10^{100}}$ and there are many more names for very large numbers that are not listed here. An example of an amount that is in the quadrillions is the number of BTU's that were used in the United States in 2002 according to the United States Department of Energy. Because it is so difficult for most people to grasp numbers this large, the names for numbers larger than quadrillion are not commonly used. Instead, large numbers are more commonly expressed in

scientific notation. Large numbers which must be written in scientific notation can be found in RSA codes, distances in outer space, and genome structure.

Length and Dimension Zero

In geometry a point is defined to have a dimension of zero while a line or line segment has dimension 1. A plane is 2-dimensional by definition.

Division by Zero: Undefined, indeterminate, or infinity

Division by zero is undefined. The reason for this is made clear by examining the inverse of division, namely, multiplication. Every division statement can be written as a corresponding multiplication statement. For example $6 \div 3 = 2$ corresponds to $2 \times 3 = 6$. Suppose we state that $5 \div 0 = 0$, then the corresponding multiplication statement gives $0 \times 0 = 5$, which is clearly a false statement. Thus, $5 \div 0 \neq 0$ and it follows that $x \div 0 \neq 0$ for any nonzero x . Now suppose that $5 \div 0 = n$ for some n . Then n must satisfy $0 \times n = 5$. However, by the multiplicative property of zero, the product of any number times zero is equal to zero. In general, if $x \div 0 = n$ then this implies that $0 \times n = x$. For nonzero x there is no value of n which makes this true, so that $x \div 0$ is undefined. If $x = 0$ we see that $0 \div 0$ is indeterminate, because $0 \times n = 0$ for any n .

It is possible to divide zero by a non-zero number. For example, $0 \div 8 = 0$ corresponds to $0 = 8 \times 0$ which is not a contradiction.

Sometimes students in upper level mathematics might be taught that a number divided by zero is infinity. This is a reasonable statement if their teacher explains that the zero in the denominator represents numbers that approach zero (and are positive). Dividing a number by

(positive) numbers which get smaller and smaller, produces values which get larger and larger and approach infinity.

Different Algebraic Expressions with Zero

It is very interesting to examine the role of zero in different algebraic expressions. The expression $\sqrt{0}$ is a valid expression because $0^2 = 0 \times 0 = 0$. Similarly the expression $\sqrt[3]{0}$ is valid because $0^3 = 0 \times 0 \times 0 = 0$. The expression x^0 is valid because any non-zero real number raised to the zero power is defined as 1. 0^0 is an interesting special case of x^0 because a lot of people will look at 0^0 and think that the answer is 0. As I learned from a website named *Ask Dr. Math*, 0^0 is said to be an indeterminate form. This means in some cases we think about it as having one value, and in other cases we say it has a different value. Setting 0^0 equal to 1 lets mathematicians use definitions in other areas of mathematics without having to make special cases for 0. A concept where having $0^0 = 1$ is useful is when using the binomial theorem. 0^x is a valid algebraic expression because the exponent x tells you how many times to write the base 0. No matter how many times you multiply zero by itself the product is always zero. Recall that $\frac{0}{0}$, or $0 \div 0$, is an indeterminate form (see previous discussion). A good example of why $\frac{0}{0}$ is an indeterminate form can be found on the *Ask Dr. Math* website. If you evaluate the limit of the expressions $\frac{x}{x}$ and $\frac{7x}{x}$ as x approaches zero, although substituting zero in for x demonstrates that both expressions approach the form $\frac{0}{0}$, the expression $\frac{x}{x}$ will approach 1 in the limit and the

expression $\frac{7x}{x}$ will have a limiting value of 7. Thus, limits which approach the form $\frac{0}{0}$ must be evaluated by alternative means since this expression does not determine a unique limit.

The Zero Discriminant and the Zero Determinant

The discriminant is a very useful tool in Algebra. The discriminant of the quadratic polynomial, $ax^2 + bx + c$, is the expression $b^2 - 4ac$, where values a , b , and c come from the polynomial. The discriminant allows the students to determine how many solutions there are to a quadratic polynomial before plugging the coefficients into the quadratic formula. The graph of a quadratic polynomial is a parabola, and the discriminant can be used to determine the number of x -intercepts of the parabola. Specifically, if the discriminant is equal to a negative number, the parabola does not cross the x -axis; if the discriminant is equal to 0, then the parabola has one x -intercept (i.e. the vertex of the parabola is on the x -axis); if the discriminant is positive, then the parabola has two, distinct x -intercepts.

It is interesting to show that shifting a parabola horizontally does not change the discriminant. To do this, we can compare the equation $y = x^2 + 0$ (a parabola that opens upwards and has its vertex at the origin) with the equation $y = (x - 3)^2 + 0$ (a parabola that opens upwards with vertex (3,0); a horizontal shift of the first equation). Evaluating the discriminant of the first parabola results in $0^2 - 4(1)(0)$, which is equal to zero. The second equation can be written as $y = x^2 - 6x + 9$, which has determinant $(-6)^2 - 4(1)(9) = 36 - 36 = 0$. Similarly, any horizontal shift of the equation $y = x^2 + 0$ results in a zero discriminant, thus the discriminant in this example is invariant under horizontal translation. However, the discriminant will vary under

vertical translations because the number of x -intercepts can change as the parabola moves up or down

The quadratic formula indicates that solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by $x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. Notice that the discriminant appears in this formula. Again, the discriminant plays an important role in evaluating the quadratic formula; if the discriminant is equal to zero, the quadratic formula reduces to $x = -b \pm 0$, so that $x = -b$ is the only solution.

A way to obtain information about two lines in the x - y coordinate plane in a system of equations is to set up a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where the entries are the parameters in the equations $ax + by = j$ and $cx + dy = k$.

Solving $ax + by = j$ and $cx + dy = k$ for y gives:

$$\begin{array}{ll} ax + by = j & cx + dy = k \\ by = -ax + j & \text{and} \quad dy = -cx + k \\ y = \frac{-ax}{b} + \frac{j}{b}, b \neq 0 & y = \frac{-cx}{d} + \frac{k}{d}, d \neq 0 \end{array}$$

If $\frac{a}{b} = \frac{c}{d}$, then the graphs of the lines determined by these two equations have the same slope,

and so are either parallel lines or the same exact line. If we consider the cross products of the

proportion $\frac{a}{b} = \frac{c}{d}$, we obtain $ad = bc$, or $ad - bc = 0$. The expression $ad - bc$ also happens to be

the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Based on the argument above, if the determinant of a

matrix whose entries are obtained from two linear equations is equal to zero, then the two lines

have the same slope. This means that the two lines are parallel (if j and k are distinct) or they are the same line (if $j = k$).

$$0! = 1$$

The fact that $0! = 1$ is a useful convention in probability. The number of ways to arrange N distinct objects is given by $N!$ and it is a convention to say that there is only one way to arrange zero objects. $1!$ equals 1 is also a useful convention in probability, it makes sense to say that there is only one way to arrange 1 object. Using something called the Gamma Function I could find an approximation for $\frac{3}{2}!$. $\frac{3}{2}! = \sqrt{\pi} \cdot \frac{1}{2} \approx 1.33$. I do not believe that there is a number such that $x! = 0$ because after learning how to find $(1.5!)$ no matter how small the fraction or decimal is it will not become zero. This is similar to the idea of limits in Calculus.

The Difference Between No Slope and a Zero Slope

Students sometimes have difficulty understanding the meaning of the number zero in the context of slope, they may have a hard time determining if the slope of the line is zero or if the line is undefined (i.e. has no slope). I explain to the students the difference between zero slope and no slope by talking about a "skier dude" in Colorado. In this example I draw skier dude skiing on a horizontal line and note that skier dude would say (in reference to the slope) "this is a total zero dude." Thus a horizontal line has a slope of zero. Then I draw a vertical line on the board with skier dude next to the top of the vertical line and ask what skier dude would say about this ski slope. By noting that skier dude would say "oh, no" (because there would be no way of surviving the slope), I make the connection that a vertical line has no slope. During this lesson I also tie together the previous knowledge of being able to have a zero in the numerator but not in

the denominator with the slope formula of rise over run. This seems to help the students understand slope a little better.

The Angle of Zero Degrees

It can be difficult for some geometry instructors to understand why there is such a thing as a zero-degree angle. It is possible that the difficulty lies in their understanding of the definition of an angle; that an angle is formed when two rays share a common endpoint. In a zero-degree angle, according to this definition, one ray is directly on top of (coincidental with) another ray; so some may consider that to just be one ray, and therefore not an angle at all. Other instructors may believe an angle is defined as the *space* between two rays (with a common endpoint), and since there is no space between the rays in a zero-degree angle, there is no angle formed. These same difficulties may arise when studying a 360-degree angle. Similarly, it can also be difficult for some to understand an angle of 180 degrees, since two rays which share a vertex and go in opposite directions form a straight line and no longer appear to be two rays. The fact that a protractor has a measure of zero on it provides evidence that there is such a thing as a zero-degree angle. A practical example of one can be considered in the firing of a projectile from one building to the top of another building with the same exact elevation. Discounting any effects from wind or gravity, the angle at which the projectile is fired would be equal to zero degrees.

Zero as a Number

The natural numbers are the numbers that are first learned when a child is growing up. They consist of the infinite set $\{1, 2, 3, \dots\}$. I tell my students that they are the numbers that

come naturally to them, like when they were learning to count on “Sesame Street”. The natural numbers are closed under addition. This means that when you add two natural numbers you get another natural number. The same rule applies when you are multiplying two natural numbers. We can extend the natural numbers to the set of whole numbers by including zero. The set of whole numbers is also closed under addition and multiplication. Whole numbers also include the zero-multiplication property; specifically when you multiply a natural number by zero we get zero for an answer. To explain this first of all it has to be understood that, when applied to whole numbers, multiplication is repeated addition. For example $4 * 3$ is the same as $4 + 4 + 4$. When you multiply $5 * 0$ you can change it to $0 * 5$ by using the commutative property of multiplication. The expression $0 * 5$ expands to $0 + 0 + 0 + 0 + 0$ which has a sum of zero. Since these steps will always work when you replace 5 with any whole number, we see that any whole number multiplied by zero will give you zero for an answer.

A general proof, assuming the additive property of zero, follows:

- | | |
|--|--|
| 1) $n + 0 = n$ | 1) Given, by the addition property of zero |
| 2) $x(n + 0) = x \bullet n$ | 2) Multiplication Property of Equality |
| 3) $x(n + 0) = x \bullet n + x \bullet 0$ | 3) Distributive Property of Equality |
| 4) $x \bullet n = x \bullet n + x \bullet 0$ | 4) Transitive Property of Equality (steps 2 and 3) |
| 5) $x \bullet n - x \bullet n = x \bullet n - x \bullet n + x \bullet 0$ | 5) Subtraction Property of Equality |
| 6) $0 = 0 + x \bullet 0$ | 6) Subtraction |
| 7) $0 = x \bullet 0$ | 7) Addition property of zero |

Thus the addition property of zero determines the multiplicative property of zero.

"Proof " that $0 = 1$

If a person does not understand all of the rules about using the number zero some very strange statements in Algebra can be “proved”. For example here is a proof that $0 = 1$ which results from a lack of understanding of the properties about the number zero:

Let $x = 0$ [Given]

Then $x(x - 1) = 0$ [Multiplication property of 0]

Therefore $x - 1 = 0$ [Divide both sides by x ; Cancellation property]

And $x = 1$ [Adding 1 to each side of the equation]

Therefore, $0 = 1$.

This “proof” violates a very important property: that division by zero is undefined. Notice that to arrive at step 3 from step 2, you have to divide by x , but since $x = 0$, this is equivalent to dividing by zero. Thus the argument is invalid.

A similar proof can be found at www.pen.k12.va.us/div/winchester/jhhs/math/humor/onezero2.html:

Given that a and b are integers such that $a = b$, prove $0 = 2$.

- | | |
|------------------------------------|--|
| 1. $a = b$ | 1. Given |
| 2. $a - b - 2 = a - b - 2$ | 2. Reflexive Property of Equality |
| 3. $a(a - b - 2) = b(a - b - 2)$ | 3. Multiplication Property of Equality |
| 4. $a^2 - ab - 2a = ab - b^2 - 2b$ | 4. Distributive Property |
| 5. $a^2 - ab = ab - b^2 - 2b + 2a$ | 5. Addition Property of Equality |
| 6. $a^2 - ab = ab + 2a - b^2 - 2b$ | 6. Associative Property of Equality |

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|-------------------------------------|--------------------------------------|
| 7. $a(a - b) = a(b + 2) - b(b + 2)$ | 7. Distributive Property |
| 8. $a(a - b) = (a - b)(b + 2)$ | 8. Distributive Property |
| 9. $a = b + 2$ | 9. Division Property of Equality |
| 10. $b = b + 2$ | 10. Transitive Property (steps 1, 9) |
| 11. Therefore $0 = 2$ | 11. Subtraction Property of Equality |

In this "proof" the error is dividing by $(a - b)$ between steps 8 and 9. The reason this is incorrect is since $a = b$, which was the given information, then $a - b = 0$. Thus the argument is incorrect.

The Zero Metaphor

There are many metaphors in society which use the word zero. One example is the "zero-balanced budget". This is a philosophy in business or in government where the amount of money taken in should be the same as the amount of money being spent. This is a concept that is very hard for groups like local and national governments to achieve; thus the big deficit we have here in the United States.

Another metaphor using the number zero is "Ground Zero". This means that an observer is at distance zero from a blast site, a phrase which was coined during the first atom bomb test in 1946. It would be interesting to find out why the Lincoln Journal Star calls their Friday entertainment section of the newspaper "Ground Zero" since there is no obvious connection between entertainment and the explosion of an atomic bomb.

In the context of measuring temperature, scientists use "absolute zero" to identify the temperature at which molecules stop moving. Although a temperature of absolute zero has never been achieved, temperatures very close to it have been reached in laboratories.

A website called "The School of Wisdom", indicates the geometrical representation for zero-dimensional object is a point. It is only a representation because physically it is impossible to draw a point with no length or width.

According to a train of thought called Tao, "The Zero Dimension is the point, the infinitely small place holder. It exists not in space, but in time only. It is the moment between past and future, the subject, zero." An article describing the Tao philosophy also stated that the Zero Dimension is the home of the natural numbers and the subject point, zero, is pure awareness.

Conclusion

The number zero is a very powerful tool in mathematics which has many different applications and rules. I have learned many truly amazing ways to think about and work with zero from writing this paper. I never realized how many different rules there were for the number zero in mathematics. This helped me realize why I have to be very careful in my teaching when the number zero is involved in the lesson. The number zero is a very special number which is much more important and powerful than what its name is associated with in our society.

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Resources

The Fourth Dimension

Wikipedia

Ask Dr. Math

School of Wisdom