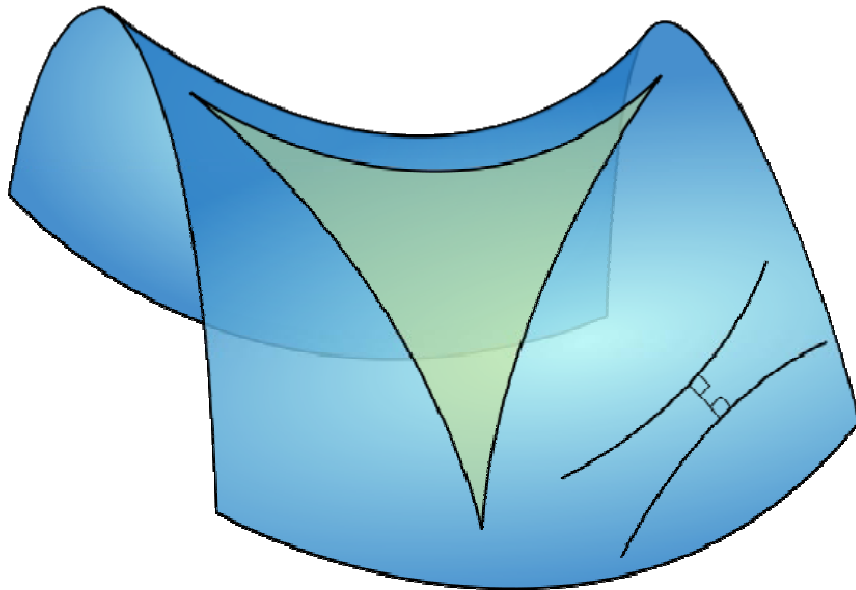


HYPERBOLIC GEOMETRY

Christina L. Sheets

July 2007



HISTORY

In general, when one refers to geometry, he or she is referring to Euclidean geometry. Euclidean geometry is the geometry with which most people are familiar. It is the geometry taught in elementary and secondary school. Euclidean geometry can be attributed to the Greek mathematician Euclid of Alexandria. His work entitled The Elements was the first to systematically discuss geometry. Since approximately 600 B.C., mathematicians have used logical reasoning to deduce mathematical ideas, and Euclid was no exception. In his book, he started by assuming a small set of axioms and definitions, and was able to prove many other theorems. Although many of his results had been stated by earlier Greek mathematicians, Euclid was the first to show how everything fit together to form a deductive and logical system.

In mathematics, geometry is generally classified into two types, Euclidean and non-Euclidean. The essential difference between Euclidean and Non-Euclidean geometry is the nature of parallel lines. Recall, that Euclid started with a small set of axioms (postulates), five to be precise. The first four of these axioms are very clear and concise.

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as a radius and one endpoint as center.
4. All right angles are congruent.

The fifth axiom, however, consistently appears to be more complex than the first four.

5. If a straight line falls on two straight lines in such a manner that the interior angles on the same side are together less than two right angles, the straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

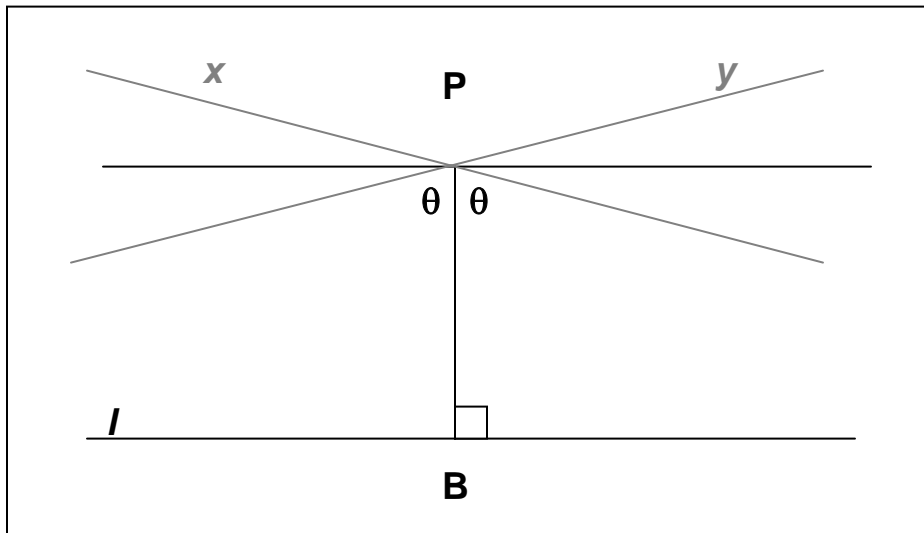
Although this axiom has since been stated in simpler forms (Example: Through a point not on a given straight line, one and only one line can be drawn that never meets the given line.), mathematicians were still troubled by its complexity, and were convinced that it could be proven as a theorem using the other four axioms. Many tried to find a proof by contradiction. They either assumed that there were at least two lines parallel or no lines parallel to the original line under the given conditions. These two tactics gave rise to two new geometries, hyperbolic (at least two lines parallel) and elliptic (no lines parallel). At the time of these discoveries, it was believed that these two geometries were false, and many people refused to believe that another geometry, other than Euclidean, existed.

In the nineteenth century, however, hyperbolic geometry was extensively studied by János Bolyai and Nikolai Ivanovich Lobachevsky. (Because of this, hyperbolic geometry is also known as Bolyai-Lobachevskian geometry.) Euclidean geometry can be modeled by a “flat plane”, and the simplest model for elliptical geometry is a sphere, but devising a model for hyperbolic geometry proved to be a difficult task. In 1868, Eugenio Beltrami showed that the pseudosphere had the appropriate curvature to model a portion of hyperbolic space, and later defined the Klein model, the Poincaré disk model, and the Poincaré half-plane model which represented the entirety of hyperbolic space. He used these models to prove that if Euclidean geometry was

logically consistent, then so was hyperbolic geometry.

LINES AND ANGLES IN HYPERBOLIC GEOMETRY

The property that sets hyperbolic geometry apart from other geometries allows for more than one line to be parallel to a given line through any one point. There are two classes of non-intersecting lines. Let B be the point on line l and let P be a point not on line l , such that the line PB is perpendicular to line l . (See picture below.) Let x be a



line through P such that x does not intersect l , and the angle theta counterclockwise from PB to x is as small as is possible without forcing any intersection to occur between line x and line l ($x \parallel l$). Line x is considered to be a hyper-parallel line. The line y that forms the same angle theta, but clockwise from PB , would also be hyper-parallel, but there can be no others. All of the other lines through point P not intersecting line l form angles larger than theta with line PB , and are considered ultra-parallel (disjointly parallel). Because there are an infinite number of angles between 0° and 90° , and each one determines two lines through P , there are an infinite number of parallel lines through P ultra-parallel to l . Therefore, in hyperbolic geometry there exists a modified

form of the parallel postulate:

In Hyperbolic Geometry, given any line l , and a point P not on l , there are two, and only two, lines through P hyper-parallel to l , and infinitely many lines through P ultra-parallel to l .

The angle of parallelism, in hyperbolic geometry, varies greatly based on the length of line PB , or the distance of point P from line l . This angle of parallelism varies with what is called the $\Pi(p)$ function, described by Nikolai Ivanovich Lobachevsky. This function produces a unique angle of parallelism for each length of line PB . Recall, the angle of parallelism in Euclidean geometry is a constant 90° , regardless of the length of line PB . In hyperbolic geometry, as the length of line PB gets shorter, the angle of parallelism will get closer and closer to 90° , but as the length of line PB gets longer, the angle of parallelism will get closer and closer to 0° . I find it interesting that the closer point P gets to line l , the more it behaves like it should in Euclidean geometry. As the plane involved gets flatter, it conforms more to the Euclidean properties.

MODELS OF THE HYPERBOLIC PLANE

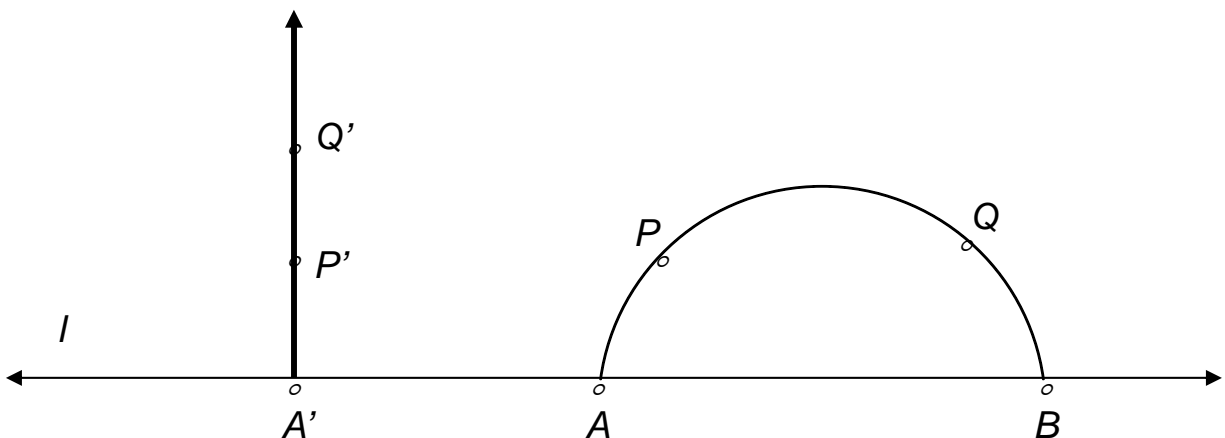
There are four commonly used models for hyperbolic geometry that define a real hyperbolic space which satisfies the axioms of hyperbolic geometry. These models are the Klein model, the Poincaré disc model, the Poincaré half-plane model, and the Lorentz model, or hyperboloid model.

- The Klein model (or projective disc model) considers the hyperbolic plane as the interior of a circle. In this model hyperbolic lines are the chords of the circle. This model is the most simplistic, however, it distorts the hyperbolic angles.

- The Poincaré disc model (or conformal disc model) also uses the interior of a circle as a plane. Lines are represented by arcs of circles that are orthogonal to the boundary circle, as well as diameters of the boundary circle. (Note: Two circles are orthogonal when their tangents at each intersection point are perpendicular.)
- The Poincaré half-plane model takes half of the Euclidean plane, as determined by a Euclidean line l , to be its hyperbolic plane. (The line l is not actually included in this plane.) Lines are either semicircles orthogonal to l or rays perpendicular to l .
- The Lorentz model (or hyperboloid model) uses a two-dimensional hyperboloid of revolution in a three-dimensional space. (This model looks a lot like a Pringle potato chip to me.) This is the most complex of the four models.

THE POINCARÉ HALF-PLANE MODEL

In order to develop the Poincaré half-plane model, one must consider a fixed line, l , in a Euclidean plane. Assume that l is on the x-axis of the Euclidean plane. **A point** in this model of the hyperbolic plane is any point on one side of line l (in this case, above). **A line** in this model of the hyperbolic plane is either a semicircle whose center is on line l , or a ray that is perpendicular to line l . (See below) Let P and Q denote two



points in this model of the hyperbolic plane. If the unique line that passes through these two points is a semicircle, then this line approaches the boundary line l in two Euclidean points, A and B , and the **distance** between points P and Q is given by the following equation.

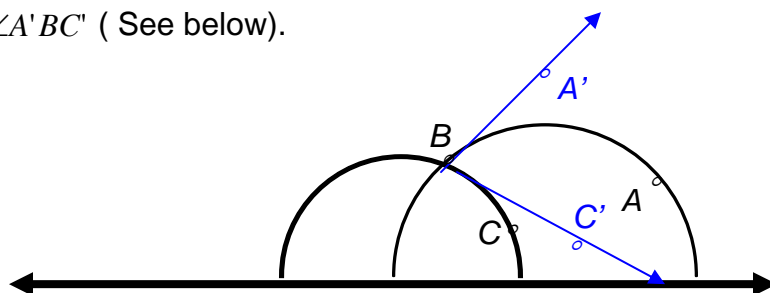
$$d(PQ) = \left| \ln \frac{|PA|/|PB|}{|QA|/|QB|} \right|, \text{ where } |PA|, \text{ etc. denotes the Euclidean distance from point } P \text{ to point } A.$$

Let P' and Q' be two other points in this model of the hyperbolic plane. If the unique line that passes through these two points is a ray, then this line approaches the boundary line l at one Euclidean point, A' , and the **distance** between points P' and Q' is given by the following equation.

$$d(P'Q') = \left| \ln \frac{|P'A'|}{|Q'A'|} \right|, \text{ where } |P'A'|, \text{ etc. denotes the Euclidean distance from point } P' \text{ to point } A'.$$

A circle in this model of the hyperbolic plane is defined as the set of all points in the plane equidistant from a given point. (This is the same as the Euclidean definition of a circle.)

Assume there are three points A , B , and C in this model of the hyperbolic plane. If one constructs the Euclidean rays BA' and BC' that are tangent to lines BA and BC at point B . **The measure of the hyperbolic angle**, $\angle ABC$, is equal to the measure of the Euclidean angle, $\angle A'BC'$ (See below).



The reason that this works is the Poincaré half-plane model is conformal, which means that the hyperbolic angles in this model are exactly the same as the Euclidean angles under the afore mentioned conditions.

HYPERBOLIC TRIANGLES IN THE POINCARÉ HALF-PLANE MODEL

Similar to Euclidean geometry a hyperbolic triangle is three distinct points connected by three (hyperbolic) line segments, any one of which connects only two of said points. Despite this fact, hyperbolic triangles are quite different from their Euclidean counterparts. Because hyperbolic line segments are usually curved, the interior angles of a hyperbolic triangle add up to strictly less than 180° , which clearly violates the triangle sum property in Euclidean geometry that says “The sum of the measures of the interior angles of a triangle is always 180° .” The amount less than 180° , or $\frac{2\pi}{2} = \pi$ radians, is called the defect. The defect function of a hyperbolic triangle is $\pi - (\alpha + \beta + \gamma)$ where angles alpha, beta, and gamma are measured in radians, and are the interior angles of a triangle. This formula was derived by Gauss when he was only seventeen years old, and it gives the area of any hyperbolic triangle. Because the sum of the angles of a hyperbolic triangle cannot be less than 0, it stands to reason that the area of a triangle cannot be greater than π . One of the most surprising facts in hyperbolic geometry is that there is an upper limit to the possible area of a triangle, but there is no boundary on the perimeter.

There are two other major differences between hyperbolic triangles and Euclidean triangles:

- The sides of a hyperbolic triangle can get as large as you want, the area

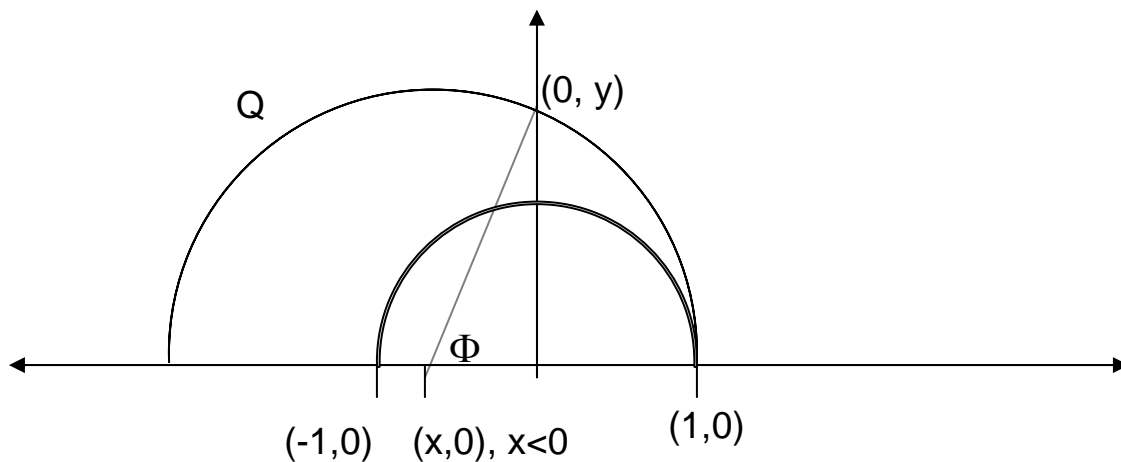
of any hyperbolic triangle is always less than π .

- Similar hyperbolic triangles do not exist. If two triangles have the same angles, they are, in fact, congruent.

Usually the points, or vertices of the triangle, are considered to be in the hyperbolic plane, but if the vertices exist on line l at infinity they are considered to be ideal. If all three points are ideal, then the triangle is said to be an ideal triangle. An ideal triangle has an angle sum of 0° , therefore, an ideal triangle has an area of π and an infinite perimeter. Any two ideal triangles are considered to be congruent. Any point within an ideal triangle is within constant distance of some two sides of the triangle.

ANGLE OF PARALLELISM

In hyperbolic geometry, the **angle of parallelism** Φ is the angle at one vertex of a right hyperbolic triangle that has two hyper-parallel sides. The angle depends on the length, a , between the right angle and the vertex to the angle of parallelism Φ . In the half-plane model of the hyperbolic plane, a relationship between a and Φ can be established. Let Q be a semicircle with its center on the x -axis that passes through $(1, 0)$ and $(0, y)$, $y > 1$. (See below) Q is tangent to the unit semicircle, so these



hyperbolic lines are parallel. The y -axis crossed both semicircles, making a right angle with the unit semicircle and an angle, Φ , with Q . The angle at the center of semicircle Q , $(x,0)$ $x < 0$, subtended by the radius to $(0, y)$ is Φ because the two angles have sides that are perpendicular. The radius of semicircle Q is $1 + x$ or $1 - x$. Based on the Pythagorean theorem, $x^2 + y^2 = (1-x)^2$, so $x = \frac{(1-y^2)}{2}$. Based on the metric of the half-plane model, let $\log y = a$, so that $y = e^a$. Now the relationship between Φ and a can be deduced from the triangle whose vertices are $(x, 0)$, $(0, 0)$, and $(0, y)$.

$$\tan \Phi = \frac{y}{-x}, \text{ but } x = \frac{(1-y^2)}{2}, \text{ so } \tan \Phi = \frac{2y}{y^2-1}.$$

$$\text{Recall that } y = e^a, \text{ so } \tan \Phi = \frac{2e^a}{e^{2a}-1}.$$

Now, using some of the hyperbolic trigonometry identities,

$$\tan \Phi = \frac{2e^a}{e^{2a}-1} = \frac{2(\cosh a + \sinh a)}{\cosh 2a + \sinh 2a - 1} = \frac{2(\cosh a + \sinh a)}{2\sinh^2 a + 1 + 2\sinh 2a - 1} = \frac{2(\cosh a + \sinh a)}{2\sinh a(\sinh a + \cosh a)} = \frac{1}{\sinh a}$$

$$\text{This means } \tan \Phi = \frac{1}{\sinh a}.$$

BIBLIOGRAPHY

- Anderson, J. W. (2005). *Hyperbolic geometry* (2nd ed.). New York: Springer.
- Castellaños, J. (1994). *NonEuclid: 8: Area*. Retrieved July 18, 2007 from <http://www.cs.unm.edu/~joel/NonEuclid/model.html>
- Castellaños, J. (1994). *NonEuclid: 10: Disk and upper half-plane models*. Retrieved July 18, 2007 from <http://www.cs.unm.edu/~joel/NonEuclid/area.html>
- Greenberg, M. J. (1993). *Euclidean and non-euclidean geometries: Development and history*. New York: W. H. Freeman.
- Reynolds, W. F. (1993). Hyperbolic geometry on a hyperboloid. *American Mathematical Monthly*, 100, 442-455. Retrieved on July 18, 2007.
- Samuels, D. (March, 2006). Knit theory. *Discover Magazine*. 27(3). Retrieved on July 18, 2007.
- Stewart, I. (2001). *Flatterland*. New York: Perseus Publishing.

**Master of Arts in Teaching (MAT)
Masters Exam**

Christina L. Sheets

In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization
in the Teaching of Middle Level Mathematics in the Department of Mathematics.

David Fowler, Advisor

July 2007