

**Master of Arts in Teaching (MAT)  
Masters Exam**

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# **Perimeter and Area of Inscribed and Circumscribed Polygons**

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This paper looks at comparing the perimeter and area of inscribed and circumscribed regular polygons. All constructions will be made with circles of radius equal to 1 unit. To begin this exploration, I created a circle with a radius of 1 (for my purposes I used 1 inch as my unit of measure). I chose my first construction to contain the most basic regular polygon, an equilateral triangle. A regular polygon implies that all sides of the figure are equal and all interior angles of the figure are congruent. My first construction shows an equilateral triangle inscribed in a circle (see Appendix A). Next, I needed to find the perimeter of this inscribed triangle. I used a ruler to measure the distance of one of the sides, then multiplied that value times 3, the number of sides of the triangle. This method will work for finding the perimeter of all regular polygons. The only change would be substituting three with the number of sides of the polygon being studied. A general formula would be: Perimeter of a regular polygon with  $n$  sides equals the length of one side times  $n$ .

Instead of using a compass and ruler to create hand constructions, I used an internet site to aid in my constructions. Using the program "GeoGebra," I was able to construct very neat and accurate pictures. Once the constructions of the inscribed and circumscribed polygon were done, I printed them out and used a ruler to measure the perimeter and area of each figure. I recorded these values in tables that I created to keep track of the calculations that I gathered throughout this exploration (see Appendices B and C).

Due to the fact that all of my constructions were made with a circle of radius 1 unit, I was able to generalize a formula for the perimeter and area of the inscribed and circumscribed triangles, squares, and hexagons. The formulas for squares were easy to find and the basis for the formulas of triangles and hexagons rested on the fact that I was able to identify a 30-60-90

degree triangle within each figure. In Math 802T we explored the characteristics of such triangles. Generally speaking, an equilateral triangle has all sides equal to length  $s$ . If you bisect one of the angles you create a 30-60-90 degree triangle. This right triangle has an unknown height of  $h$ , a hypotenuse of  $s$ , and a base of length  $s/2$  (see Appendix D). For my specific right triangle, the hypotenuse is of length 1 and the base is  $1/2$ . Then, I used the Pythagorean Theorem to calculate the height.

$$h^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$h^2 + \frac{1}{4} = 1$$

$$h^2 = \frac{3}{4}$$

$$h = \sqrt{\frac{3}{4}}$$

$$h = \frac{\sqrt{3}}{2}$$

Finally, trying to find perimeter and area formulas for the octagons presented a unique challenge because I was not able to identify any 30-60-90 triangles. Instead, I located isosceles triangles. For the inscribed and circumscribed octagons, I relied on trigonometry functions that I learned in Math 896 to find the measure of the side lengths. Additionally, at this stage in my project I felt it was appropriate to introduce the general formulas that can be used to find the perimeter of any regular inscribed or circumscribed  $n$ -gon in relation to a circle with a diameter of 1 unit. Since my circles had a radius of 1 unit I had to double the values obtained using the following formulas.

Inscribed  $n$ -gon

$$h = n \tan\left(\frac{180^\circ}{n}\right)$$

Circumscribed  $n$ -gon

$$h = n \sin\left(\frac{180^\circ}{n}\right)$$

Up until this point in the project I had looked at the perimeter and area of circumscribed and inscribed regular polygons. I gathered measurements generated with my ruler, and formulas devised to find more exact values. Using a ruler to measure the side lengths of each polygon and then completing perimeter and area calculations would result in approximations due to human error. This method would result in less accurate calculations than those obtained using an appropriate formula.

Comparing all of the data I collected, it seems that with both my hand measurements and computer calculations the values of the perimeter for each figure appear to be closing in on a specific value. If I continue to calculate the perimeter of regular polygons and increase the number of sides the polygon has, the resulting values appear to be leveling off close to 3.14 (see Appendix B). This number contains the first three digits in the value  $\pi$ . This is what I expect because of the following ideas about the circumference of circles.

$$\begin{aligned} \text{Let radius } r &= 1 \\ C &= \pi 2r \text{ or } C = \pi d \\ C &= \pi(2)(1) \\ \frac{C}{2} &= \pi \end{aligned}$$

Circumference is a measure of the distance around a circle, just as the perimeter is a measure of the distance around a polygon. Thus, I can use the idea shown in the equation above and divide each of my perimeter values by two. The result should be close to the value  $\pi$ . Of course,  $\pi$  is an irrational number so any finite decimal expansion cannot be the exact value, thus my quotients will only be an approximation of  $\pi$ , i.e., approximately, 3.14.

The inscribed polygons will generate an underestimate of  $\pi$ . Circumscribed polygons will result in an overestimate of  $\pi$ . As the number of sides of the polygons increases, the

calculated values will get closer and closer to the actual value of  $\pi$ . As shown in the constructions of all my polygons (see Appendix A), as the number of sides of the polygon increases, the perimeter of that polygon gets closer and closer to the circumference of the circle. A polygon with many sides begins to resemble the circle itself, and thus the measurement of its perimeter will be very close to the circumference of the circle.

The following examples model this with perimeter values for the triangle and octagon. Notice the triangle values are considerable under and over estimates of  $\pi$ . However increasing the number of sides in the polygon from three to eight really starts to show how the values are closing in on  $\pi$ .

$$\begin{array}{l} \textit{Inscribed Triangle} \\ \frac{5.196}{2} = 2.598 \end{array}$$

$$\begin{array}{l} \textit{Circumscribed Triangle} \\ \frac{10.392}{2} = 5.196 \end{array}$$

$$\begin{array}{l} \textit{Inscribed Octagon} \\ \frac{6.112}{2} = 3.056 \end{array}$$

$$\begin{array}{l} \textit{Circumscribed Octagon} \\ \frac{6.624}{2} = 3.312 \end{array}$$

Now I will examine how the *area* of a circle is affected when the radius is equal to one.

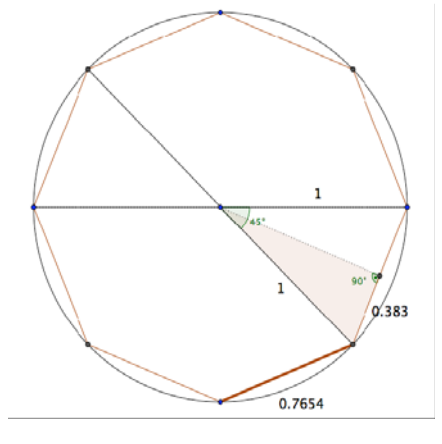
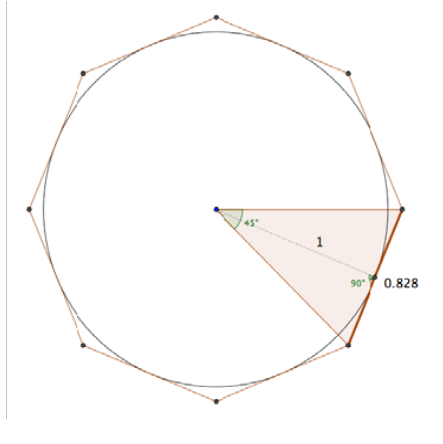
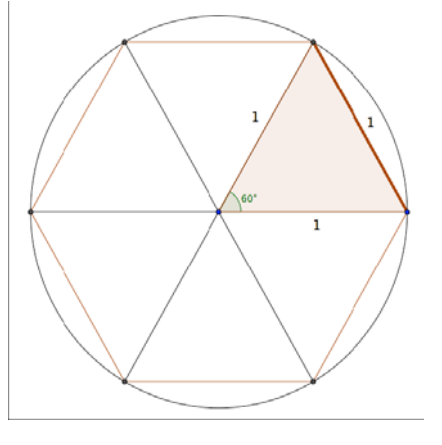
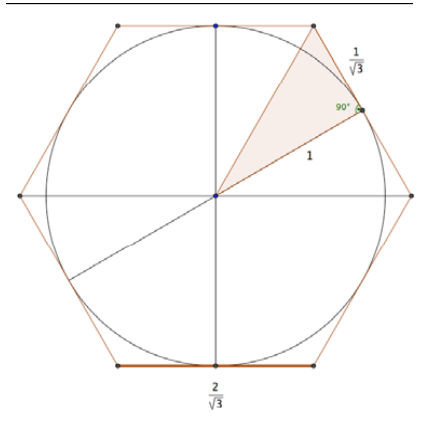
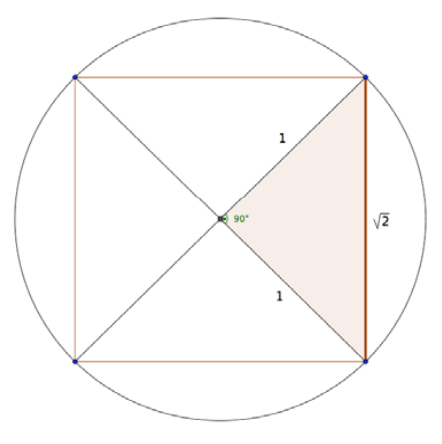
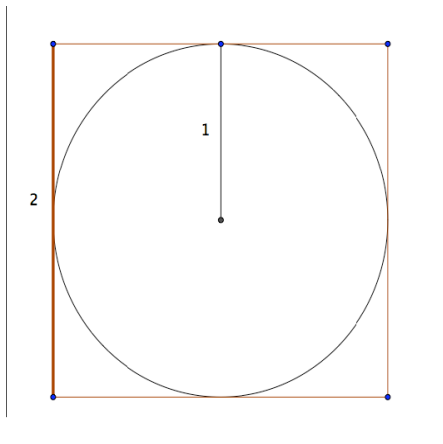
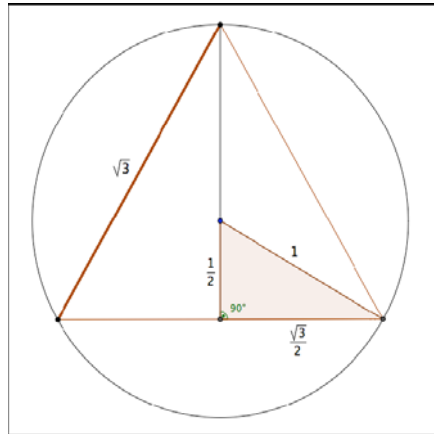
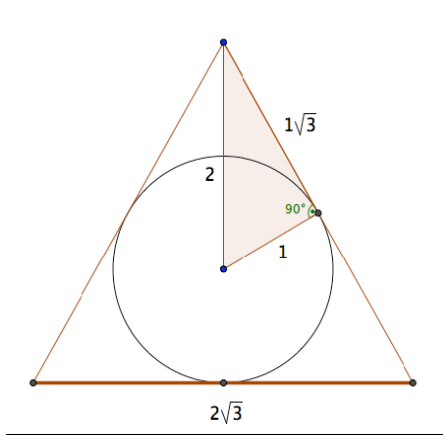
$$\begin{array}{l} \text{Let radius } r = 1 \\ A = \pi r^2 \\ A = \pi(1)^2 \\ A = \pi * 1 \\ A = \pi \end{array}$$

This equation is a reminder that the area of a unit circle will be equal to  $\pi$ . Using inscribed and circumscribed polygons, I found the area of each shape and compared these values

to  $\pi$ . Similar to my perimeter findings, when the number of sides of the regular polygon increased, the area of the figure got closer to  $\pi$ . Again, the area of the inscribed polygons represent an underestimate of  $\pi$  (the circle's area) and the area of the circumscribed polygons represent an overestimate of  $\pi$ . See Appendix C for additional area calculations and comparisons.

In summary, I have discovered that when I inscribe or circumscribe a regular polygon in reference to a circle of radius 1 unit, the perimeter of the polygon divided by 2 results in a quotient that is an approximation of  $\pi$ . As the number of sides on the polygon increases, the approximation gets closer to the value of  $\pi$ . Inscribed polygons represent an underestimate of  $\pi$ . Circumscribed polygons represent an overestimate of  $\pi$ . The same idea is true when studying the area of polygons. The area of the polygon closes in on the value of  $\pi$  as the number of sides on the polygon increases.

## **Appendix A**



## Appendix B

Perimeter calculations for inscribed and circumscribed regular polygons.

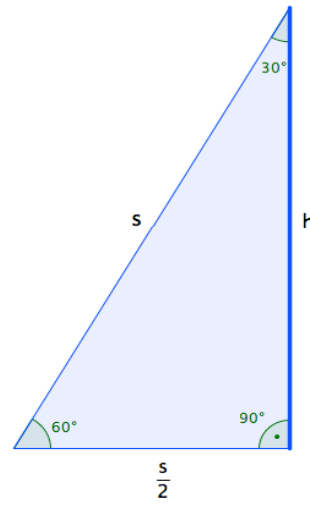
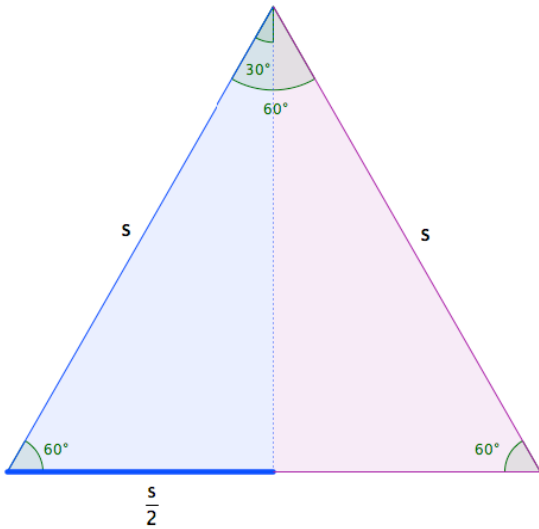
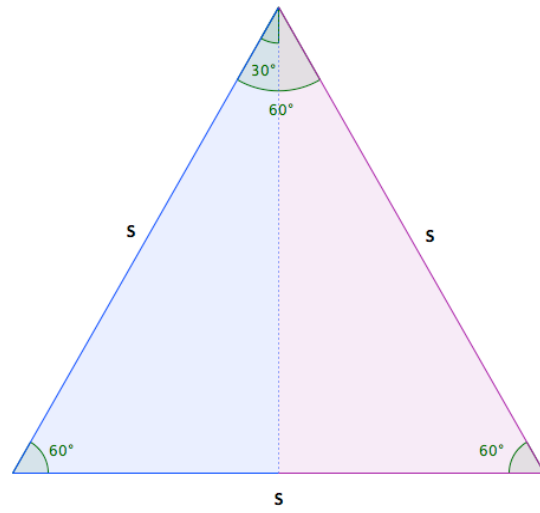
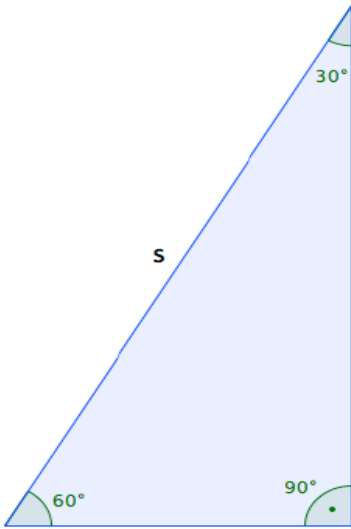
	Measurement	Calculated	Formula	Approx to $\pi$
Ins. Triangle	5.25	5.196	$3\sqrt{3}$	<b>2.598</b>
Ins. Square	5.6	5.657	$4\sqrt{2}$	2.829
Ins. Hexagon	6	6	6	3
Ins. Octagon	6.4	6.112		3.056
Ins. 20-gon		6.257		3.129
Ins. 100-gon		6.282		<b>3.141</b>
	Measurement	Calculated	Formula	Approx to $\pi$
Cir. Triangle	10.5	10.392	$6\sqrt{3}$	<b>5.196</b>
Cir. Square	8	8	8	4
Cir. Hexagon	67.2	6.928	$4\sqrt{3}$	3.464
Cir. Octagon	6.5	6.624		3.312
Cir. 20-gon		6.335		3.168
Cir. 100-gon		6.285		<b>3.143</b>

## Appendix C

Area calculations for inscribed and circumscribed regular polygons.

	Measurement	Formula	Approx to $\pi$
Ins. Triangle	1.3	$\frac{\sqrt{3}}{2}$	0.866
Ins. Square	1.96	2	2
Ins. Hexagon	2.6	$\frac{3\sqrt{3}}{2}$	2.6
Ins. Octagon	2.9		<b>3.062</b>
	Measurement	Formula	Approx to $\pi$
Cir. Triangle	2.25	$3\sqrt{3}$	5.196
Cir. Square	4	4	4
Cir. Hexagon	3.6	$2\sqrt{3}$	3.464
Cir. Octagon	3.5		<b>3.312</b>

# Appendix D



$$s^2 = h^2 + \left(\frac{s}{2}\right)^2$$

