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The Four Numbers Game

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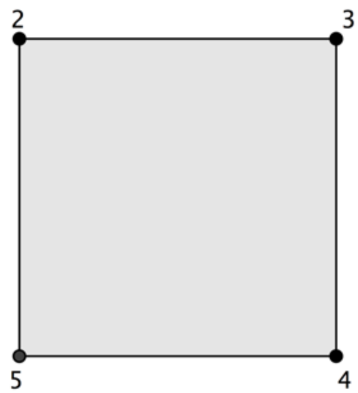
Abstract

The Four Numbers Game is a fun way to work with subtraction and ordering of numbers. While trying to find an end to a game that is played with whole numbers, there are several items that will be investigated along the way. First, we offer an introduction to how the game is played. Second, rotations and reflections of a square will be presented which will create a generalized form. Third, we explain how even and odd number combinations will always end in even numbers within four subtraction rounds. Fourth, we argue that the length of the game does not change if multiples of the original numbers are used to create a new game. Fifth, we show that all Four Numbers Games will come to an end. Sixth, we offer an investigation of the general form and some special cases and how they can help predict a more accurate end. Finally, there will be an example of how this problem could be used in a sixth grade classroom.

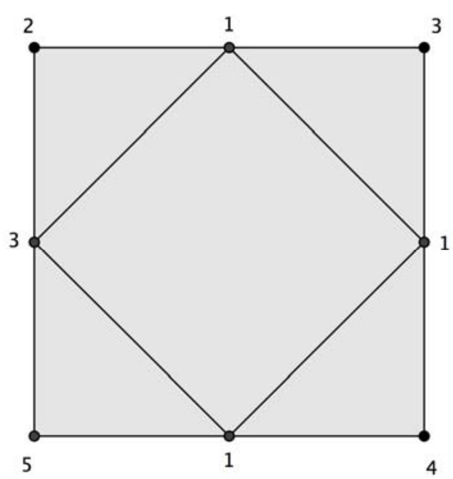
The Four Numbers Game

The Four Numbers Game begins with a square. At each of the vertices a whole number is placed.

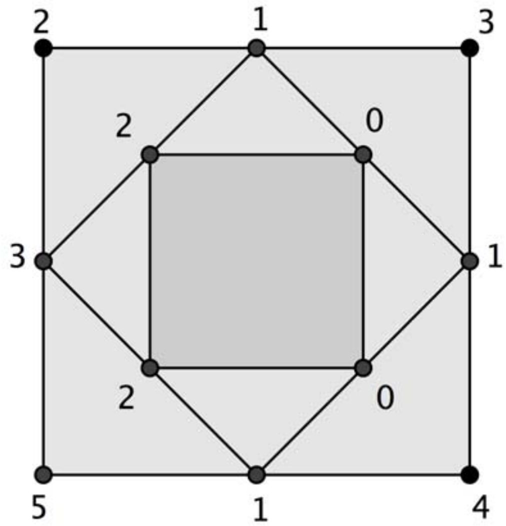


Start: 2, 3, 4, 5

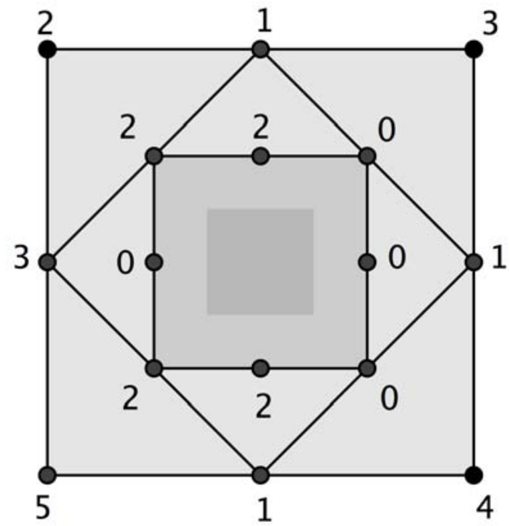
The positive difference of each pair of vertices is placed at the mid point of the corresponding side and these four points are used to create a new square. Using these four vertices (and numbers) the game is repeated. The game is repeated until you get a difference of zero on all four sides, thus finishing the game.



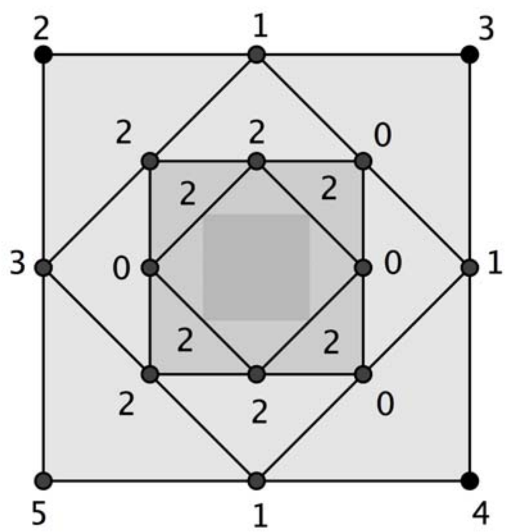
Round 1: 3-2=1, 4-3=1, 5-4=1, 5-2=3



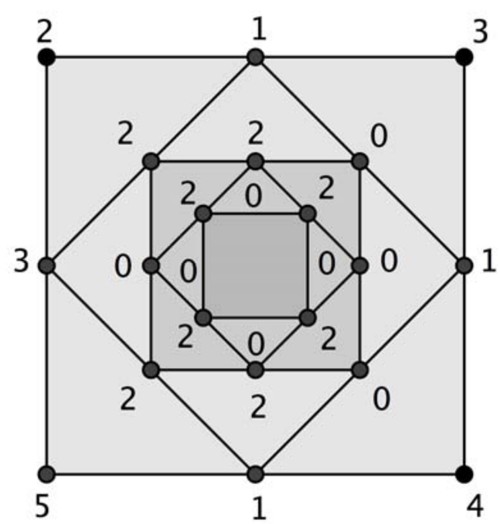
Round 2: 1-1=0, 1-1=0, 3-1=2, 3-1=2



Round 3: 2-0=2, 0-0=0, 2-0=0, 2-2=0



Round 4: 2-0=2, 2-0=2, 2-0=2, 2-0=2



Round 5: 2-2=0, 2-2=0, 2-2=0, 2-2=0

This game beginning with 2, 3, 4 and 5 took five rounds to reach zero. This will be referred to as having a *length of five* for the rest of the paper. As one can see the squares are wonderful to look at, but from this point on, a table will be better suited for "playing" the game for our purposes.

When using only single digit whole numbers, 0 to 9, it is possible for a game to have a length of eight before it finishes. The example below uses the numbers 9, 4, 1, 0. Thinking of the previous square, one takes the difference of the two numbers that are next to each other (this corresponds to consecutive vertices), using the absolute value as needed to get a whole number. I.e. when we "play" the "game" using a table, each entry is the positive difference between the number above the entry and the number in the next column to the right. Our convention is that the 1st column is the one "to the right" of the last column.

Start:	9	4	1	0
Round 1:	9-4=5	4-1=3	1-0=1	9-0=9
Round 2:	5-3=2	3-1=2	9-1=8	9-5=4
Round 3:	2-2=0	8-2=6	8-4=4	4-2=2
Round 4:	6-0=6	6-4=2	4-2=2	2-0=2
Round 5:	6-2=4	2-2=0	2-2=0	6-2=4
Round 6:	4-0=4	0-0=0	4-0=4	4-4=0
Round 7:	4-0=4	4-0=4	4-0=4	4-0=4
Round 8:	4-4=0	4-4=0	4-4=0	4-4=0

If the possible number span is from 0-44, one can find a game of length twelve using the numbers 44, 24, 13, 7. Tribonacci numbers are used here, possibly allowing the furthest inconsistent distance between four numbers in this range. Note that Tribonacci numbers are a sequence of numbers that start with 0, 0, 1, 1, 2, 4, 7, 13, 24, 44… One adds the first three numbers to get the fourth number (0+0+1=1). The next three numbers are then added together to give the fifth number (0+1+1=2) and so on.

Start:	44	24	13	7
Round 1:	20	11	6	37
Round 2:	9	5	31	17
Round 3:	4	26	14	8
Round 4:	22	12	6	4
Round 5:	10	6	2	18
Round 6:	4	4	16	8
Round 7:	0	12	8	4
Round 8:	12	4	4	4

Round 9:	8	0	0	8
Round 10:	8	0	8	0
Round 11:	8	8	8	8
Round 12:	0	0	0	0

While we do not have a proof of this, it appears that eight is the maximum length of a game that begins with single digit numbers and twelve is the maximum length for a game that begins with numbers up to (and including) 44.

In order to generalize our analysis of this game, we note that while rotating the square results in an adjusted arrangement of the vertices, it does not change the numbers which appear in the game nor does it change the length of the game. Recall the illustration representing the five rounds of the game displayed in the original example:

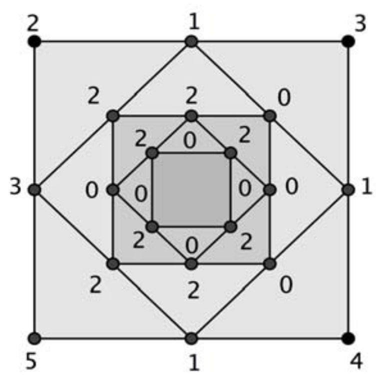
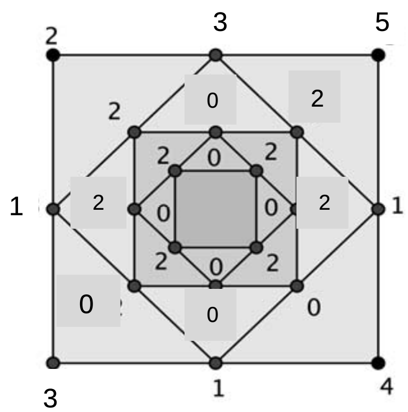
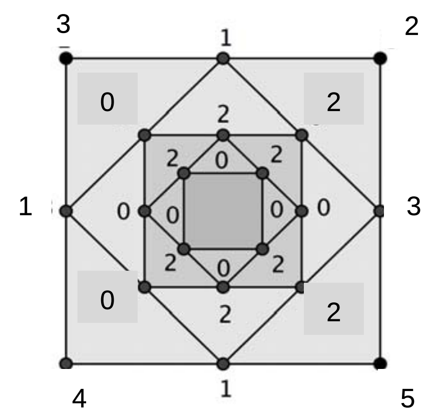


Figure 1

While the illustration represents one particular game, clearly the length of *any* Four Number game will stay the same if the square is rotated by multiples of 90°.

We also consider the effects of reflecting a square upon the length of a game. Again, consider the illustration above representing the five rounds of the game beginning with 2, 3, 4, 5. Reflecting the image about a diagonal (in this case the diagonal connecting the vertices labeled 2 and 4) and reflecting the image about a vertical line of symmetry, respectively, results in the two following figures:

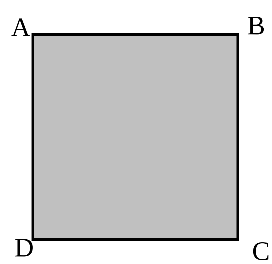




Reflection of Figure 1 about the diagonal 2-4.

Reflection of Figure 1 about a vertical line of symmetry

Again, the number of squares inside the original square does not change, therefore the length of the game will not change. Since this will hold true for any of the four lines of symmetry within the square, reflecting the square does not affect the length of the game. Thus in much of the remaining analysis, we will refer to a generalized Four Numbers game as A,B,C,D labeled as shown:



An interesting observation about the Four Numbers Game is that if a Four Numbers game has a length of at least four, then all the numbers appearing from Round 4 onward are even. Because reflections and rotations do not change the length of a game, we only need to consider these cases for the initial numbers:

E,E,E,O
O,O,O,O
E,O,E,O
E,E, O,O
O,O,O,E

Then, the steps in the game for each of these cases are shown or imbedded in the following tables:

Start:	even	even	even	odd
Round 1:	even-even=even	even-even=even	even-odd=odd	odd-even=odd
Round 2:	even-even=even	even-odd=odd	odd-odd=even	odd-even=odd
Round 3:	even-odd=odd	even-odd=odd	even-odd=odd	even-odd=odd
Round 4:	odd-odd=even	odd-odd=even	odd-odd=even	odd-odd=even

Start:	odd	odd	odd	even
Round 1:	odd-odd=even	odd-odd=even	odd-even=odd	even-odd=odd
Round 2:	even-even=even	even-odd=odd	odd-odd=even	odd-even=odd
Round 3:	even-odd=odd	odd-even=odd	even-odd=odd	odd-even=odd
Round 4:	odd-odd=even	odd-odd=even	odd-odd=even	odd-odd=even

Another useful fact in analyzing the game is that taking the initial numbers and multiplying them by a positive integer *n* does not change the length of the game. Consider, for example, a game where two entries on opposite vertices are the same. Symbolically we assume that A=C but make no other assumptions about which entries are the largest. Later in this paper we will need the fact that this game happens to have length four.

	A	B	C	D
Start:	A	B	A	D
Round 1:	|A-B|	|A-B|	|A-D|	|A-D|
Round 2:	0	|B-D|	0	|B-D|
Round 3:	|B-D|	|B-D|	|B-D|	|B-D|
Round 4:	0	0	0	0

Any game created by multiplying each entry of the game (in which A=C) by an integer $n > 0$ will also have length four.

	A	B	A	D
Start:	*n**A	*n**B	*n**A	*n**D
Round 1:	*n*|A-B|	*n*|A-B|	*n*|A-D|	*n*|A-D|
Round 2:	0	*n*|B-D|	0	*n*|B-D|
Round 3:	*n*|B-D|	*n*|B-D|	*n*|B-D|	*n*|B-D|
Round 4:	0	0	0	0

In a similar way, if we take *any* game and multiply the four initial numbers by a positive integer *n*, the k^{th} row of the new game will be *n* times the k^{th} row of the original game. Thus the two games will have the same length.

Starting with any four non-negative integers at the vertices of the square, if the largest number is not at the upper left corner (position A), the square can be rotated 90°, 180°, or 270° in order to put it there. Then, reflecting about diagonal AC, if necessary, it can be assumed that the four initial numbers A, B, C, D satisfy A≥B≥D and A≥C. This will be known as the *standard form* for the game. As we shall see, the size of C relative to B and D is what determines the maximum number of rounds needed to play the game. Without loss of generality, we will assume A≥B≥D, and A≥C (i.e., the standard form) for all cases described in the remainder of this paper.

We now address the question ‘Do all Four Numbers Games beginning with whole numbers have a finite length?’ We argue that the answer is yes.

Beginning with a game in standard form, we know that there is always a power of two that will be larger than the number A, i.e. $A < 2^k$ for some positive integer *k*. Specifically, the number A is always less than 2^A. Additionally, there will be a *least* positive power of 2 which is greater than A. Our goal is to show that taking the least possible *k* such that $A < 2^k$, and multiplying it by four provides an upper bound for the length of the game. For example, if a square started with the largest number of 9 in the A position, then $9 < 2^4$. Our theorem asserts that 4*4 or 16 is an upper bound for the length of the game. This tells us the length of the game is finite, but it does not give us the *least* upper bound for the length of the game. (We have previously claimed that 8 is the maximum length for a game where the largest number is nine.)

Consider a Four Numbers Game beginning with whole numbers, at least one of which is non-zero. Suppose that A is the largest of these vertices. Let k be the least positive integer such that $A<2^k$. If the length of the game is less than or equal to four, there is nothing to prove as the game is certainly of finite length. If the length of the game is greater than four, we follow a proof in which the key idea is given in the example below:

Start:	149	81	44	24
Round 1:	68	37	20	125
Round 2:	31	17	105	57
Round 3:	14	88	48	26
Round 4:	74	40	22	12

Since in round four all numbers are even, we can take divide the numbers at each of the vertices by two and use them to start a new game. The length of the new game (27, 20, 11, 6 in this example) will be the same as the length of the game beginning with vertices obtained in round four (74, 40, 22, 12 in the example above) since each of these values is twice the value of the corresponding vertex in the new game (i.e. the 27, 20, 11, 6 game). The length of our original game will be 4 more than the length of the revised game (i.e. the 27, 20, 11, 6 game). Thus our original game has length 4 plus the length of the new game.

We proceed:

Start:	37	20	11	6
Round 1:	17	9	5	31
Round 2:	8	4	26	14
Round 3:	4	22	12	6
Round 4:	18	10	6	2

Again we begin a new game with initial values half of those obtained in round 4 since they are all even. We now know the length of the original game is at least 8.

Continuing the example:

Start:	9	5	3	1
Round 1:	4	2	2	8
Round 2:	2	0	6	4
Round 3:	2	6	2	2

Notice that in round three of the example above, there are (at least) two vertices with equivalent values. As we demonstrated previously, a game which two values are equal will have four steps. Therefore, since 4 + 4 + 3 + 4 = 15, our original game has a length of 15. The claim made by our theorem is more modest. Since $149<2^8$, the theorem claims that the length of the game is less than 4*8=32. In other words thirty-two is an upper bound, but not the *least* upper bound.

In general, when we determine that $A<2^k$, we can see that every four rounds of the game we can replace the numbers by a new game with the maximum number being less than 2^k divided by 2, or 2^{k-1}. Eventually, we are playing a game in which all entries are less than 2. Such a game takes at most 4 rounds. Thus, in at least 4*k* steps, we must get to the end of our original game. Therefore we have shown that all Four Numbers Games beginning with whole numbers have finite length.

We now know that every Four Number game has a finite length and we have an upper bound for the length of each game. It is helpful to examine certain circumstances that lead to the determination of a better bound for a Four Numbers game. Looking at the standard form, there are three different cases that might occur.

Case 1: A≥C≥B≥D
Case 2: A≥B≥D≥C
Case 3: A≥B≥C≥D

Case 1: A≥C≥B≥D

We want to show that all Case 1 games have lengths four or less. Note first that if A=C and B=D and A≠B, then the game ends in two rounds as shown in the following table. (If A=B the length is one.)

	A	B	C	D
Start:	A	B	A	B
Round 1:	A-B	A-B	A-B	A-B
Round 2:	0	0	0	0

Now let's begin a general Case 1 game. Note that after two rounds, we have the first and third entries equal and the second and fourth entries equal. Thus the game will end in at most two more rounds (see table below).

	A	B	C	D
Start:	A	B	C	D
Round 1:	A-B	C-B	C-D	A-D
Round 2:	A-C	B-D	A-C	B-D

Case 2: A≥B≥D≥C

We want to show that all Case 2 games have lengths six or less. Note first that if A=B and D=C then the game has length three or less.

	A	B	C	D
Start:	A	A	D	D
Round 1:	0	A-D	0	A-D
Round 2:	A-D	A-D	A-D	A-D
Round 3:	0	0	0	0

Now let's begin a general Case 2 game. Note that after two rounds, we have the second and fourth entries equal. Previously we demonstrated that any game with two opposite vertices equal has length four. Thus the game will end in four more rounds or less.

	A	B	C	D
Start:	A	B	C	D
Round 1:	A-B	B-C	D-C	A-D
Round 2:	|(A-B)-(B-C)|	B-D	|(A-D)-(D-C)|	B-D

Starting with Case 3: A≥B≥C≥D

While all Four Number Games have finite length, it is in Case 3 that one can find games that have length greater than 6. In fact, there is no universal bound on the length of all games. Even in Case 3, we are able to find conditions that lead to better bounds in the length of certain games. For example, if A-B=C-D, the game has length of five. Also, if A=B, B>C>D the game will have length six or less.

Note that after the first round of the game A-B=C-D, so that the first and third entries are equal. Thus the game will end in four more rounds or less.

A=B, B>C>D	A	B	C	D
Start:	A	B	C	D
Round 1:	A-B	B-C	C-D=A-B	A-D

Note that after the second round of the game A=B, B>C>D, we have the first and third entries equal. Thus this game will end in four more rounds or less and the game will have length six or less.

	A	B	C	D
Start:	A	A	C	D
Round 1:	0	A-C	C-D	A-D
Round 2:	A-C	$\lvert(A-C)-(C-D)\rvert$	A-C	A-D

Lesson Plan

Teaching this lesson to a sixth grade classroom would be a great way to investigate mathematical vocabulary, computation fluency, number sense through absolute value, investigation of general form, and how reflections and rotations affect the general form. Students could be given a square with four positive integers in the corners and allowed to solve for the solution together as a class. There could also be some squares on the paper permitting them to create their own game using whole numbers 0 to 9.

They would then be grouped randomly and permitted to check with their group to see if anyone got a length different than the one presented together in class. There would be two more squares they could use on the sheet to see if they could come up with different lengths than what the members in the group got. A hint would be on the board that states “8 sure is great!”

The next day, randomly grouped students would be permitted to check with each other on the different game lengths they got. The teacher would ask students if anyone got a length of 0, 1, 2, 3, 4, 5, 6, 7, or 8 and place those examples on the board. Once there was an example of each one, or as many as the class could come up with, they will be directed to think about the square in terms of A, B, C, D.

The teacher would have a moveable square on the board labeled A, B, C, and D. She/he would also have a square already with the length of four completed as well. The teacher would then rotate the length 4 squares 90°. She would ask the students if the length of the game of the rotated square is the same, allowing students to debate, if needed. When a decision is made that it did not change, write on the board that 4 is equal to the length of the game A,B,C,D, and to the length of the game, B,C,D,A. Then rotate the just completed game of length four another 90°. Again, ask if the length of the game would stay the same. The students might be ready to answer yes, the length would stay the same. Write on the board, each of the four games, A,B,C,D and B,C,D,A and C,D,A,B, and D,A,B,C has the same length. Give the students time to reflect on this thought. Allow them to write down what they are thinking.

The next day have students share why these squares all shared the same game length. The idea being that although the box was turned on its side, it never lost the number of squares inside of it. Moving on to reflection, the class would work together to talk about the reflection of the box. The students would then be permitted to see if the length of the game changed. They would be asked to explain that for the next day of class.

Throughout this experience the students will investigate number theory by exploring the differences between even and odd numbers. They will also be asked to make conjectures about the generalized standard form of the Four Numbers Games for different lengths 0 through 4. They will be permitted to prove their conjectures through classroom discussions, models, and examples. The students would be asked to share what they realize about square A,B,C,D on the board.

The Four Numbers Game is a fun way to work with subtraction and ordering of numbers. While trying to find an end to a game that is played with whole numbers, there are several items that were demonstrated. After an introduction to how the game was played, rotations and reflections of a square were presented to create a generalized form. An interesting explanation of even and odd number combinations resulted in even numbers within four subtraction rounds. The length of the game did not change when a multiple of the initial numbers were used to begin a game. All Four Numbers Games were shown to have finite length by investigating the standard form and its special cases. Finally, there was an example of how this problem could be used in a sixth grade classroom.

References

Sally, Judith D., Sally, Paul J., *TriMathlon: A Workout Beyond the School Curriculum.* 2003. A. K. Peters Ltd. Natick, MA.