The Art Gallery Question
Expository Paper

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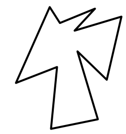
<u>MAT question</u>

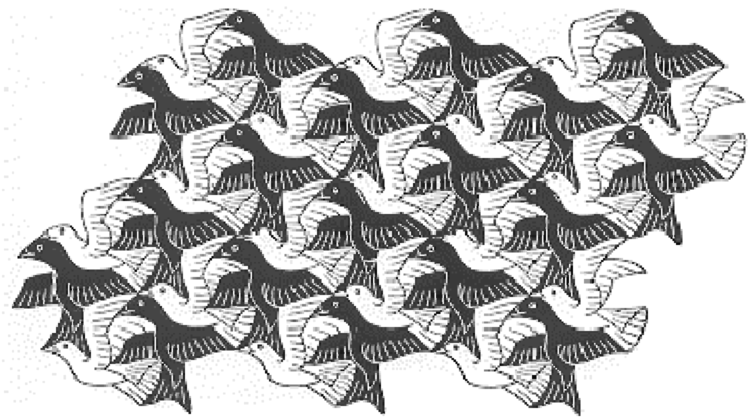
Suppose you have an arbitrary room in an art gallery with v corners, and you wanted to set up a security system consisting of cameras placed at some of the corners so that each point in the room can be seen by one of the cameras. How many cameras do we need? (See the example at right for a possible room with an interesting shape.)



In the early 1900's a graphic artist by the name of Paul Klee was studying art, most of which involved geometry. He spent his career drawing and etching unique combinations of angles and colors. By the end of his career he had over 10,000 paintings and pieces of art that expressed his love of color and geometry. He often wondered how the angles and color were connected.

Victor Klee, not a relation to Paul Klee, was just starting his life as a mathematician as Paul Klee was coming to the end of his career. Victor Klee also liked geometry and art, and was drawn to Paul Klee's work. Victor studied Paul Klee's work and found that Paul was not only interested in art and geometry but also interested in infinite sums. Both of Paul's interests drew Victor further into the studies of geometry and the idea of 'infinite sums'. As Victor continued to study mathematics his questions regarding geometry and infinite sums led to the branch of mathematics known as computational geometry. During his long career as a mathematician, Victor Klee has made contributions to a wide variety of mathematics, such as discrete and computational geometry, convexity, combinatorics, graph theory, functional analysis, mathematical programming and optimization, and theoretical computer science.

In the beginning of Victor Klee's career when he was exploring geometry he became friends with M. C. Escher. Escher is another famous graphic artist. His art was known for geometric graphics and many were infinite in manner. In my seventh grade classes we discuss tessellations which are an important part of Escher's work and many times infinite designs. While Escher had no formal education in either math or science, he was respected by both mathematicians and scientists. Here is an example of one of M. C. Escher's tessellations.



During the beginning of Victor Klee's career he often wrote to Escher discussing mathematics, geometry and infinite patterns. Escher was much older than Victor Klee but they still enjoyed discussing art and infinite patterns. Escher and Klee made references in their correspondence to Paul Klee's work of angles and color. With the discussion of Paul Klee's work and Escher's work this caused Victor to wonder once again about their art and the geometric angles that they were famous for. By the early 1970's Victor Klee was well on his way to becoming a knowledgeable professor of infinite sums and mathematics. At this time Victor posed a question to his fellow mathematicians regarding an 'arbitrary polygonal closed curve with v vertices, and how many vertices were required so that every point on the interior of the curve would be visible'. (Burger & Starbird, 2000, p 221)

Vάclav Chvatal another young mathematician took this question as a challenge and began to work on a solution. Chvatal's solution was conceptually simple but he includes several special cases in his proof.

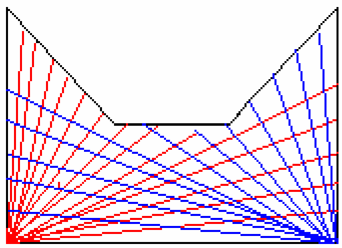
Chvatal's problem was this:

> *Given a simple n-gon, what is the minimum number of vertices from which it is possible to view every point in the interior of the polygon?*
> *(Answer: [n/3].)*

(A polygon is *simple* if it has no self-intersections. The sides of such a polygon may only meet at their endpoints, and never more than two at a time.)

If the polygon is convex, its whole interior can be viewed from any one vertex. In general, this is of course not true. The question is not about a particular polygon, but about the entire family of simple polygons: what is the worst case scenario? What is the minimum number of vertices that would answer the question for every admissible n-gon? [n/3] is the answer. (n = the number of vertices)

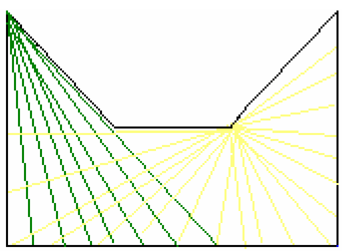
Construct an example (think of a comb with n triangular teeth) where the required number of vertices is exactly [n/3].

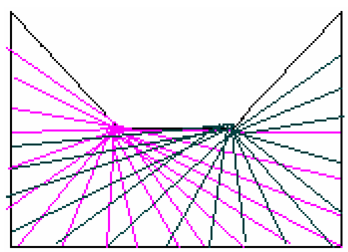


There are 6 vertices so 6/3 = 2

Look at the other examples, the vertices chosen were vertices that cameras could be placed. The vertices chosen were the only two are needed to view every point inside the polygon in this example. We could have chosen two different vertices but we would have to be able to view all areas inside the polygon.

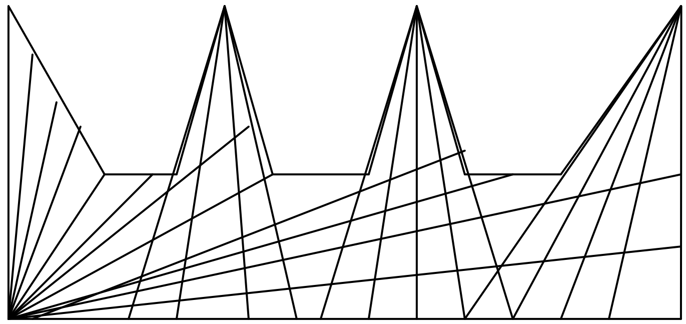
There are 6 vertices, so 6/3 = 2 needed viewing spots





Many of the examples in my research used a comb shaped polygon. Each of these comb shapes has a point tine (like a comb tooth). If we start with a simple comb, two tines, then try a three tine comb for our testing examples, we can start simple. Now, if we think of other polygons and include the idea of how many cameras would be needed to view art work, look at the possibilities using the 'tine' shape.

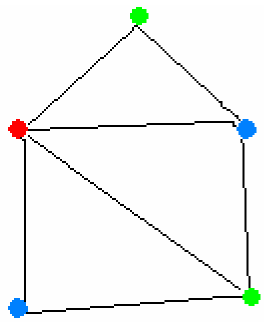
4 tine shape	12 vertices	12/3 = 4 cameras(at most)
5 tine shape	15 vertices	15/3 = 5 cameras(at most)
6 tine shape	18 vertices	18/3 = 6 cameras(at most)



Chvatal's proof took any polygon and followed the next steps to prove that his ideas work. The idea is to create a polygon triangulation by drawing its diagonals. When finished, i.e., after the maximum possible number of the (nonintersecting) diagonals have been drawn. Displayed is one of the possible groups of corners from which the whole interior of the polygon is observable.

The proof proceeds in a few steps:

1. Triangulate the polygon with its diagonals.
2. Show that for such a diagonal triangulation of the polygon, its vertices can be colored with three colors, such that all three colors are present in every triangle of the triangulation.

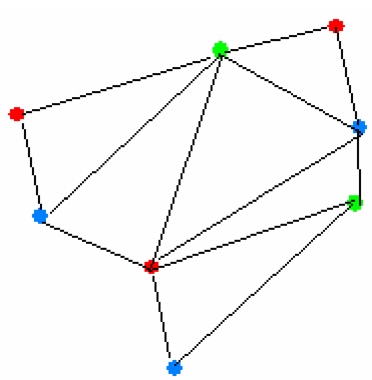


Choose the vertices of the polygon assigned the least frequent color.

Starting from the end, assume a diagonal triangulation has been found satisfying 1 and 2. For example, assume there are B blue vertices, R red vertices and G green vertices. Then obviously B + R + G = n. One of the numbers does not exceed the other two.

Without loss of generality we assume that

$$B \leq R \text{ and } B \leq G.$$



Then:

$$3 \cdot B \leq B + R + G = n,$$

so that B $\leq$ n/3, and, since B is a whole number,

$$B \leq [n/3].$$

*Remember you are free to pick the colors of the first two adjacent vertices but after that the colors of the rest of the vertices are already determined.

Now, still going backwards, given a diagonal triangulation, how do we color the vertices to satisfy #2? Note that, in a diagonal triangulation, the sides of any triangle are either sides or the diagonals of the polygon, and at most 2 may be diagonals. The triangles could only abut each other at the diagonals, not the sides of the polygon. Furthermore, only two triangles in a triangulation may share the same diagonal. It follows that each triangle in a triangulation is adjacent two 1 or 2 other triangles. The neighboring triangles share a side and, consequently, two vertices.

Start with any triangle of the triangulation and color its vertices with three colors in a random fashion. Its neighbor triangles already have two vertices assigned two different colors; the requirements of #2 force a unique color on the remaining vertex. Continue in this manner with 1 or, as the case may be, 2 adjacent triangles until all vertices have been assigned a color. The process stops when on either end we run into a triangle composed of 2 sides and 1 diagonal of the polygon.

The diagonal triangulation is always possible. The proof is by induction. Any diagonal that lies entirely in the interior of a polygon splits the latter into two with fewer sides. Find such a diagonal and then apply this same consideration to both constituent polygons until only triangles remain. A triangle, i.e., a 3-gon, is the only element of its only triangulation. The proof of #1 seems thus to be complete. However, it is based on the premise that any simple polygon has at least one diagonal that lies entirely in the interior of the polygon.

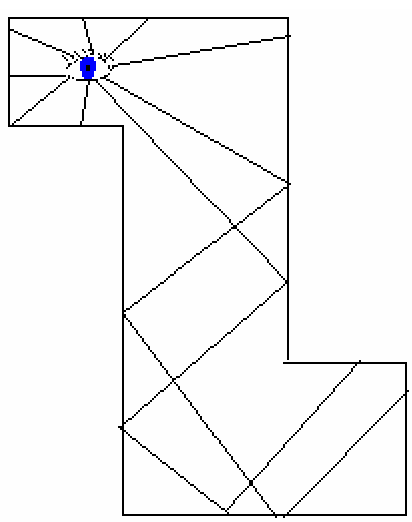
More simply put, after you count the vertices of each color you understand that each color can’t occur more than 1/3 of the time. (Burger & Starbird, 2000, p 221)

After mathematicians had seen Chvatal's solution another mathematician Steve Fisk, from Bowdoin College, found Chvatal's proof unappealing. He continued to think about this question. He was traveling with his wife in Afghanistan and was stuffed in a crowded hot bus trying to sleep, when his solution came to him. Fisk is known for a more simplified version of this proof. [v/3] is always enough for any camera to see every point inside a polygon.

More simply put the 'Art Gallery' question is just to convert this to a counting question. Victor Klee used this counting method to base the beginning of combinatorial geometry. In this branch of mathematics, geometry questions are answered by counting techniques.

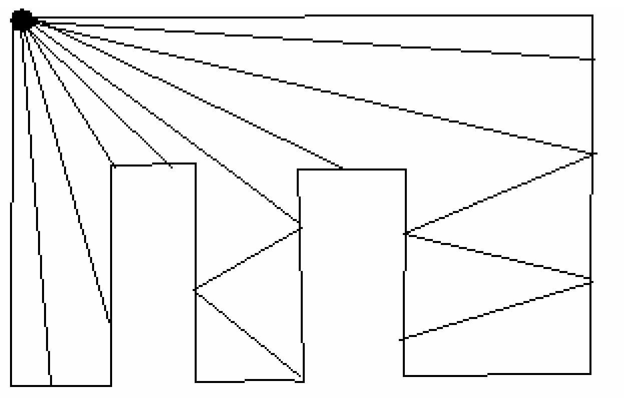
As teachers of middle school mathematics we are familiar with famous mathematics and mathematicians like Newton, Euler and Pythagoras. We've since learned about other very important mathematicians like Leibniz creating calculus, and Fermat and his theorem of primes. Although we might think of most theorems being discovered hundreds of years ago, there is new math being proved and discovered continuously. With the world constantly advancing, especially in technology, there is a need for new mathematical knowledge. This is how the proof of Chvatal's became necessary because of the advancing of technology. If we continue to think about the art gallery question and add mirrors to the problem, it changes the question slightly but requires more thought. Note: at this point in time, no one knows the answer to this slightly changed question.

This other variation of this theorem is stated like this. Assume the art gallery is some polygon (closed curve) but the walls are mirrors, if they reflect and you could see around corners, how many guards would be needed to view or guard the whole gallery?



We might be able to see everything, all points inside the gallery by using mirrors. It is thought, by many, that every point would be visible, if a guard would be at a certain point.

Here is one example of how the guard might see using mirrors.



Although this drawing makes it look like only one guard would be needed, how many reflections are too many? When does the distance become too much? How can we prove visibility always exists?

All of these questions just bring more questions. If the camera is at the height of the ceiling, what about the blind spot directly below the camera? If the camera is moved out away from the wall, say one foot, the blind spot is still below the camera but is now larger. If the camera rotates, the camera creates a blind spot when it moves, the area not being viewed becomes the blind spot. When does the distance of vision become too far?

So with the solution of the art gallery question comes more questions. With technology advancing at huge rates, cameras are large, small, come with wide angle lens, telephoto lens and simple mirrors could all be studied to answer the question of current gallery guard questions. This leads us to the next proof, when will it be solved?

Visibility within a house of mirrors question.

For an arbitrary polygon closed curve having mirrored sides, must there exist a point from which every point inside the polygonal curve is visible?

Quick final note:

Computational geometry is a branch of mathematics that Victor Klee has become well known for. On further investigation this is a study of problems that are solved and stated using geometry. This form of mathematics is a part of computer graphics, computer aided designs and CAD/CAM (a computer program used in the high schools). Computational geometry is also used in robotics (motion planning and visibility problems), GPS (global positioning systems), circuits, and numerically controlled machines. These are just a few examples to show that this is a form of new mathematics that has become extremely important to our everyday lives.

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Math in the Middle

Math in the Middle has been an extraordinary experience. As an experienced teacher I've enhanced my teaching skills, deepened my math knowledge, and made some wonderful friends and connections in the teaching world. These last two years have been enlightening, stressful, challenging, time consuming, and just plain fun at times.

My new friends and colleagues come from many different cities across the state of Nebraska. They teach in different types of schools from small to large, elementary to high school, they teach math but many also teach other subjects. This adds to the wide range of skills that we all drew from during the time in Math in the Middle. We've had math lessons that range from high school physics to elementary cut and paste. They were all exciting and new. Our group of teachers worked well with one another. We learned to work as a group and collaborate on many projects and drew from the skills of everyone.

The professors and graduate students that were our teachers during this time were patient and knowledgeable throughout our learning process. They were willing to support us during the ups and downs of our learning. We worked long hours to ensure our understanding of the new material and they were right beside us being our support. I would like to think of them as our teaching colleagues also.

One of the most important lessons I've learned is how to better solve problems. I saw many different ways to problem solve. I've learned new and different methods of problem solving from my colleagues. Some teachers made lists, while others listed out the givens and sorted thru the material slowly, and some drew pictures. Every one of these methods was enlightening and added to my problem solving skills. I feel confident that I could at least sort thru a problem and eventually solve most middle school problems.

Problems solving was probably my weakest area in middle school math. While working with students that have lower reading skills I was not always willing to tackle the problem solving and the reading issues on a regular basis. During this time we were given many opportunities to teach in front of our colleagues. As I watched each of these teachers, I gathered more techniques that I can add to my repertoire of skills. With my new confidence and skills that I've developed I am a much better teacher than before Math in the Middle.

During this grant program I've become better acquainted with the NCTM and state standards. I have a good understanding of the process standards that are the basis of our Lincoln, NE, math curriculum. I have a better understanding of how and why our district has chosen our textbooks and materials. With this knowledge I can enhance my lessons to make better use of our curriculum.

The participants of Math in the Middle come from different communities that follow different curriculums, have different guidelines and have different NCLB testing practices. Throughout these two years we've had many opportunities when the teachers have spoken about their districts, their curriculum and these NCLB testing processes. While my job does not allow me to participate in many of the decisions made by our district, it is interesting to listen to the teachers of smaller communities discuss all the work they have to go through to have their students prepared for the NCLB testing.

While I will never be an author, I do feel that I've become a better writer. The action research project gave me an insight to how important research has become. During this learning process we read articles that were written by classroom teachers that were very interesting. It was important to me to read articles by teachers who were in the classroom. Classrooms have changed dramatically during the last five years which makes these readings relevant to me. Too many times articles are written by people who have been away from the classroom for an extended amount of time which affects how relevant an article could be.

The research project gave me many opportunities to practice writing, and express my ideas and opinions. Throughout Math in the Middle, my writing has improved and made my thinking processes more organized. I was required to write about math problems, explain the different steps, and make sure the fine details were all there. Our writings ranged from opinions to technical math problem information. Thank goodness for computers. If we were still using typewriters we would all still be typing away somewhere.

All of our learning has been wonderful, but the bottom line has to be the students. We will be better teachers and stay better teachers because of our connections thru Math in the Middle. We've developed materials like the Habits of Mind problems that we can draw upon to enhance our lessons and stimulate the students learning. So at the end of the next school year we will all know that each of us has put our heart and soul into our teaching so we can help our students the best they can be in mathematics.

While I've listed multiple skills that Math in the Middle has helped me develop, being treated as a professional was an added bonus. We were given opportunities to attend conferences, we were interviewed by important people, treated to dinners, and made to feel extra special in so many ways during these past two years. With these skills, I hope we teachers can repay you all by fulfilling the expectations that you have of us.