Fluency with Fractions

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PEANUTS by Charles M. Schulz

HOW CAN YOU MULTIPLY 4½ BY 6 5/8 ?! THAT'S RIDICULOUS!

WHY SHOULD I LEARN THAT? I'LL BET IN ALL MY LIFE I'LL NEVER MULTIPLY 4½ BY 6 5/8 !!

WHAT MAKES YOU THINK SO?

I'LL REFUSE TO DO IT!
A Quick Question

$\frac{7}{5}$ or $1 \frac{2}{5}$?
Many U.S. students have a weak understanding of fractions

• 2004 NAEP - 50% of 8th-graders could not order three fractions from least to greatest (NCTM, 2007)
• 2004 NAEP, Fewer than 30% of 17-year-olds correctly translated 0.029 as 29/1000 (Kloosterman, 2010)
• One-on-one controlled experiment tests - when asked which of two decimals, 0.274 and 0.83 is greater, most 5th- and 6th-graders choose 0.274 (Rittle-Johnson, Siegler, and Alibali, 2001)
It is possible to have good number sense for whole numbers, but not for fractions...

Sowder, J. and Schappelle, Eds., 1989

Why?
Many students (and some teachers) don’t think of fractions as numbers.

• Teacher A: The same fraction can represent different quantities.
  – For example, one quarter of a large pizza is more than one quarter of a small pizza.

Would any teacher feel they need to use class time to point out that two large pizzas are more than two small pizzas?
Fractions are Foundational

In its report, the National Math Panel said the failure to master fractions was for American students the greatest obstacle to learning algebra.

The catchword for math teachers today should be “fractions,” said Skip Fennell, former president of the National Council of Teachers of Mathematics.
How do you think about fractions?

My advice for elementary teachers:
• A fraction is a number.
• Develop the number line.
• Be flexible.
• The goal is to understand the rational numbers.
Your recipe for pancakes calls for \( \frac{4}{5} \) of a pint of milk but you only have \( \frac{3}{5} \) of a pint of milk. What fraction of a batch of pancakes can you make if you have enough of the other ingredients?

Your recipe for pancakes calls for 4 pints of milk but you only have 3 pints of milk. What fraction of a batch of pancakes can you make if you have enough of the other ingredients?
What is a Fraction?*

- Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into $b$ equal parts; understand a fraction $a/b$ as the quantity formed by $a$ parts of size $1/b$.

* CCSS – 3.NF.1
Who is right?

Nell believes the picture represents the fraction $\frac{5}{3}$ while Estelle believes it represents $\frac{5}{6}$. 
What is a Fraction?*

• Understand a fraction as a number on the number line; represent fractions on a number line diagram.
  – Represent the fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into $b$ equal parts. Recognize that each part has size $1/b$ and that the right endpoint of the part based at 0 locates the number $1/b$ on the number line.
  – Represent a fraction $a/b$ on a number line diagram by marking off $a$ lengths $1/b$ from 0. Recognize that the resulting interval has size $a/b$ and that its endpoint locates the number $a/b$ on the number line.

* CCSS – 3.NF.2
Panel

Robert Siegler (Chair)
Carnegie Mellon University

Thomas Carpenter
University of Wisconsin-Madison

Francis (Skip) Fennell
McDaniel College

David Geary
University of Missouri at Columbia

James Lewis
University of Nebraska-Lincoln

Yukari Okamoto
University of California-Santa Barbara

Laurie Thompson
Elementary Teacher

Jonathan (Jon) Wray
Howard County (MD) Public School

Project Officer - Susan Sanchez
Institution of Education Sciences (IES)
Proficient fourth graders should have a conceptual understanding of fractions and decimals…”

(NCES, 2009, p. 18).
Facets of the lack of student conceptual understanding... just a few

• Not viewing fractions as numbers at all, but rather as meaningless symbols that need to be manipulated in arbitrary ways to produce answers that satisfy a teacher

• Focusing on numerators and denominators as separate numbers rather than thinking of the fraction as a single number

• Confusing properties of fractions with those of whole numbers

IES – Practice Guide - Fractions, 2010 (pp. 6-7)
Fractions Practice Guide authors concluded:

• “A high percentage of U.S. students lack conceptual understanding of fractions, even after studying fractions for several years; this, in turn, limits students’ ability to solve problems with fractions and to learn and apply computational procedures involving fractions.”

IES – Practice Guide - Fractions, 2010 (pp. 6-7)
Fractions Practice Guide Recommendations

1. Build on students’ **informal understanding of sharing and proportionality** to develop initial fraction concepts.

2. Help students recognize that **fractions are numbers and that they expand the number system beyond whole numbers**. Use **number lines as a central representational tool** in teaching this and other fraction concepts from the early grades onward.

3. Help students **understand why procedures for computations with fractions makes sense**.

4. Develop students’ **conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems**.

5. Professional development programs should place a high priority on improving teachers’ **understanding of fractions and how to teach them**.
Recommendation 1

Build on students’ informal understanding of sharing and proportionality to develop initial fraction concepts.

– Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.

– Extend equal-sharing activities to develop students’ understanding of ordering and equivalence of fractions.

– Build on students’ informal understanding to develop more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.
Fair share:
Another Interpretation

"How come PJ got four sandwiches and I only got two?"
Recommendation 1 Questions

Would you rather share your favorite pizza with 3 other people or with 7 other people? (McNamara and Shaughnessy, *Beyond Pizzas and Pies*, 2010)

If four friends had 13 cookies to share, how many cookies would each person get? Would this be *more* or *less* than if you shared 10 cookies?
Recommendation 2

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

- Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share.
- Provide opportunities for students to locate and compare fractions on number lines.
- Use number lines to improve students’ understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.
- Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students’ ability to translate among these forms.
1) Draw a number line and show where to place the fraction $\frac{9}{5}$. Explain your thinking.

\[ \frac{9}{5} \] is equivalent to $1 \frac{4}{5}$ and is almost 2 so it has to go there.

2) Order from smallest to greatest: $\frac{7}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{9}{8}$.

\[ \frac{3}{8} \quad \frac{5}{8} \quad \frac{7}{8} \quad \frac{9}{8} \]

\[ \frac{1}{8} = 1.25 \]
1) Draw a number line and show where to place the fraction $\frac{9}{5}$. Explain your thinking.

Because $\frac{9}{5}$ is a top heavy fraction, it was said it was $1 \frac{4}{5}$. $1 \frac{4}{5}$ is right behind 2 on the number line.

2) Order from smallest to greatest: $\frac{7}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{9}{8}$.

\[
\begin{array}{cccc}
\frac{3}{8} & \frac{5}{8} & \frac{7}{8} & \frac{9}{8}
\end{array}
\]

3) Order from smallest to greatest: $\frac{3}{5}$, $\frac{3}{7}$, $\frac{3}{4}$, and $\frac{3}{8}$.

\[
\begin{array}{cccc}
\frac{3}{8} & \frac{3}{4} & \frac{3}{7} & \frac{3}{5}
\end{array}
\]
Recommendation 2 Problems

• The line segment below has length 9/8 of a unit. Subdivide the line segment to create a line segment of length ¾ of a unit. Explain how you know your answer is correct.

• What reasoning helps you place the following fractions on the number line below?

\[
\begin{align*}
\frac{15}{47} & \quad \frac{4}{9} < \frac{49}{99} < \frac{44}{99} \\
\frac{1}{3} & \quad \frac{1}{2} \\
\frac{2}{3} & \quad \frac{31}{45} < \frac{21}{29} < \frac{31}{29} \\
\frac{49}{52} & \quad \frac{27}{99} \\
\frac{21}{45} & \quad \frac{31}{49} \\
\end{align*}
\]
Recommendation 3

Help students understand why procedures for computations with fractions makes sense.

- Use area models, number lines, and other visual representations to improve students’ understanding of formal computational procedures.
- Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.
- Address common misconceptions regarding computational procedures with fractions.
- Present real-world contexts with plausible numbers for problems that involve computing with fractions.
Estimate the sum of the three fractions on a number line.

\[ S = \frac{5}{8} + \frac{4}{7} + \frac{2}{5} \]

Student: I know that \( \frac{5}{8} \) and \( \frac{4}{7} \) are each a bit more than \( \frac{1}{2} \) and \( \frac{2}{5} \) is a bit less than \( \frac{1}{2} \) so the sum is close to \( 1 \frac{1}{2} \).

Teacher: Is it smaller or larger than \( 1 \frac{1}{2} \)?

Student: Larger because \( \frac{1}{10} < \frac{1}{8} \).
What Happens Here?

<table>
<thead>
<tr>
<th>Expression</th>
<th>Comparison</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} \times \frac{3}{4}$</td>
<td>$&lt;$ or $&gt;$</td>
<td>$\frac{3}{4}$</td>
</tr>
<tr>
<td>$\frac{3}{4} \times \frac{1}{2}$</td>
<td>$&lt;$ or $&gt;$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{1}{2} \div \frac{3}{4}$</td>
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<td>$\frac{3}{4}$</td>
</tr>
</tbody>
</table>
Now what?

• There are 25 students in our class. Each student will get \( \frac{1}{4} \) of a pizza. How many pizzas should we order.

How many pizzas should we order?

Fractions!
Try asking the same question using whole numbers.

• There are 25 students in our class. Each student will get 3 pizzas. How many pizzas should we order?

• How many pizzas should we order?
Task 1

There are 25 students in our class. Each student will get $\frac{1}{4}$ of a pizza. Your job is to decide how many pizzas we should order? Be sure to show your work.

How many pizzas should we order? 7
Recommendation 3 Problems

- Use a number line to show your solution to the following problem. Nita walked 1/2 mile on Monday, Wednesday, and Friday. On Saturday she walked 3/4 of a mile. What was her mileage for the 4 days?
Recommendation 4

Develop students’ understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

– Develop students’ understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students’ developing strategies for solving ratio, rate, and proportion problems.

– Encourage students to use visual representations to solve ratio, rate, and proportion problems.

– Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.
Recommendation 4 Problem

- It takes $1\frac{1}{2}$ bags of cement mix to pave $\frac{2}{5}$ of your driveway.
  - How many bags of cement mix are needed to pave the entire driveway?
  - What fraction of the driveway can be paved with only one bag of cement mix?
Recommendation 4 Problem

• It takes you \( \frac{3}{4} \) of a minute to run \( \frac{2}{3} \) of a lap on the track. Running at the same pace,
  – How long will it take you to run one lap?
  – How far can you run on the track in one minute?
Recommendation 5

Professional development programs should place a high priority on improving teachers’ understanding of fractions and of how to teach them.

– Build teachers’ depth of understanding of fractions and computational procedures involving fractions.
– Prepare teachers to use varied pictorial and concrete representations of fractions and fraction operations.
– Develop teachers’ ability to assess students’ understandings and misunderstandings of fractions.
Recommendation 5 Problem

Find two fractions between 3/5 and 5/8 using two different approaches. Do not use decimals.

Common Numerators

Common Denominators

Average

Basketball Average
Recommendation 5 Problem

Find two fractions between $\frac{3}{5}$ and $\frac{5}{8}$ using two **different** approaches. Do not use decimals.

**Common Numerators**

\[
\frac{3}{5} = \frac{15}{25} = \frac{30}{50} < \frac{30}{49} < \frac{30}{48} = \frac{15}{24} = \frac{5}{8}
\]

**Common Denominators**

**Average**

**Basketball Average**
Recommendation 5 Problem

Find two fractions between 3/5 and 5/8 using two different approaches. Do not use decimals.

Common Numerators
\[
\frac{3}{5} = \frac{15}{25} = \frac{30}{50} < \frac{30}{49} < \frac{30}{48} = \frac{15}{24} = \frac{5}{8}
\]

Common Denominators
\[
\frac{3}{5} = \frac{24}{40} = \frac{48}{80} < \frac{49}{80} < \frac{50}{80} = \frac{25}{40} = \frac{5}{8}
\]

Average

Basketball Average
Recommendation 5 Problem

Find two fractions between $\frac{3}{5}$ and $\frac{5}{8}$ using two different approaches. Do not use decimals.

Common Numerators

$$\frac{3}{5} = \frac{15}{25} = \frac{30}{50} < \frac{30}{49} < \frac{30}{48} = \frac{15}{24} = \frac{5}{8}$$

Common Denominators

$$\frac{3}{5} = \frac{24}{40} = \frac{48}{80} < \frac{49}{80} < \frac{50}{80} = \frac{25}{40} = \frac{5}{8}$$

Average

$$\left(\frac{3}{5} + \frac{5}{8}\right) \times \frac{1}{2} = \frac{24 + 25}{5 \times 8 \times 2} = \frac{49}{80}$$

Basketball Average
Recommendation 5 Problem

Find two fractions between $\frac{3}{5}$ and $\frac{5}{8}$ using two different approaches. Do not use decimals.

**Common Numerators**

\[
\frac{3}{5} = \frac{15}{25} = \frac{30}{50} < \frac{30}{49} < \frac{30}{48} = \frac{15}{24} = \frac{5}{8}
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**Common Denominators**

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\frac{3}{5} = \frac{24}{40} = \frac{48}{80} < \frac{49}{80} < \frac{50}{80} = \frac{25}{40} = \frac{5}{8}
\]

**Average**

\[
\left(\frac{3}{5} + \frac{5}{8}\right) \ast \frac{1}{2} = \frac{24 + 25}{5 \ast 8 \ast 2} = \frac{49}{80}
\]

**Basketball Average**

\[
\frac{3}{5} < \frac{3 + 5}{5 + 8} = \frac{8}{13} < \frac{5}{8}
\]
Definition: \( \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \) where \( a, b \neq 0, c \neq 0 \) are whole numbers.

\[
\begin{align*}
\frac{2}{3} \times \frac{4}{5} & \quad \frac{2}{3} \times \frac{6}{5} & \quad \frac{2}{3} \times \frac{3}{5} \\
\frac{2}{3} \times \frac{8}{5} & \quad \frac{2}{3} \times \frac{60}{5} & \quad \frac{2}{3} \times \frac{3}{2} \\
\frac{2}{3} \times \frac{1\frac{4}{5}}{5} & \\
\end{align*}
\]
Be Flexible

\[ \frac{2}{3} \times \frac{4}{5} \quad \frac{a}{b} \cdot \frac{c}{d} \quad \frac{2}{3} \times \frac{6}{5} \quad \frac{a}{b} \cdot \frac{kb}{d} \quad \frac{2}{3} \times \frac{3}{5} \quad \frac{a}{b} \cdot \frac{b}{d} \]

\[ \frac{2}{3} \times 8 \quad \frac{a}{b} \cdot c \quad \frac{2}{3} \times 60 \quad \frac{a}{b} \cdot \frac{kb}{d} \quad \frac{2}{3} \times \frac{3}{2} \quad \frac{a}{b} \cdot \frac{b}{a} \]

\[ \frac{2}{3} \times 1 \frac{4}{5} \quad \frac{a}{b} \cdot \left( n + \frac{c}{d} \right) \]